Borel complexity of graph homomorphism

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Graph homomorphism

Given a countable first-language $\mathcal{L}$, let $\mathrm{Mod}_{\mathcal{L}}$ be the Polish space of countable $\mathcal{L}$-structures.

We will mostly be interested in the space $\mathcal{G}$ of countable undirected graphs, which is a Polish subspace of $\mathrm{Mod}_{\mathcal{L}_g}$ with $\mathcal{L}_g$ consisting of a single binary relation symbol. For notational simplicity, given an undirected graph $G \in \mathcal{G}$ we denote by G again its edge relation.

We consider the binary relations $\preccurlyeq$ (**homomorphism**), $\sqsubseteq$ (**embeddability**) and $\cong$ (**isomorphism**) on $\mathcal{G}$, where $h \colon G_1 \to G_2$ is

- a homomorphism if $v \; G_1 \; w \Rightarrow h(v) \; G_2 \; h(w)$
- an embedding if it is injective and $v \; G_1 \; w \iff h(v) \; G_2 \; h(w)$
- an isomorphism if it is a surjective embedding.

Main goal

Determine the complexity of the classification problem on $\mathcal{G}$ up to **homomorphic equivalence** $\approx$, where $G_1 \approx G_2$ iff $G_1 \preccurlyeq G_2 \preccurlyeq G_1$.

Borel reducibility

$\preccurlyeq$ and $\sqsubseteq$ are analytic quasi-orders, while $\cong$ and $\approx$ are analytic equivalence relations, hence we can use Borel reducibility to study them.

Definition

Given analytic binary relations R and S on standard Borel spaces X and Y, respectively, we set $R \leq_B S$ iff there is a Borel map $f \colon X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1 \; R \; x_2 \iff f(x_1) \; S \; f(x_2).$$

Intended meaning: S is at least as complex as R.

We write $R \sim_B S$ if $R \leq_B S \leq_B R$.

Our starting point

Louveau and Rosendal proved

Theorem (Louveau-Rosendal, 2005)

1. The embeddability relation $\sqsubseteq$ is **complete** for analytic quasi-orders, i.e., $R \leq_B \sqsubseteq$ for every analytic quasi-order R.
2. Also the homomorphism relation $\preccurlyeq$ is complete for analytic quasi-orders.

In particular, countable graphs cannot be classified up to $\approx$.

Part 1 was later extended to

Theorem (S. Friedman-M., 2011)

The embeddability relation $\sqsubseteq$ is in fact **invariantly universal**, i.e. for every analytic qo R there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C} \subseteq \mathcal{G}$ s.t. $R \sim_B \sqsubseteq\restriction \mathcal{C}$.

Some natural questions

Problem 1

Let $\mathcal{C}$ be some natural/interesting class of countable graphs. How complex is $\preccurlyeq\restriction \mathcal{C}$? Can we classify elements of $\mathcal{C}$ up to $\approx$?

Obstacles: The graphs in the Louveau-Rosendal construction are very special: for example, they contain arbitrary large cliques (and this is essential in the argument!).

Problem 2

Is $\preccurlyeq$ invariantly universal? What about its restrictions $\preccurlyeq\restriction \mathcal{C}$ to classes $\mathcal{C} \subseteq \mathcal{G}$ as in Problem 1?

Obstacles: The only known technique to prove invariant universality of $\preccurlyeq$ needs a very "rigid" Borel reduction from $\sqsubseteq$ to $\preccurlyeq$, which is not what is proved in the Louveau-Rosendal theorem; we need a different proof.

Example 1: controlling chromatic number, (odd) girth, etc...

Girth: $\gamma(G) =$ length of the shortest cycle in G if there is any, and otherwise $\gamma(G) = \infty$

Odd girth: $\gamma_o(G) =$ length of the shortest cycle with odd length if there is any, and $\gamma_o(G) = \infty$ otherwise

Chromatic number: $\chi(G) =$ smallest $n \leq \aleph_0$ for which there is $c \colon G \to n$ such that $c(v) \neq c(w)$ whenever $v \; G \; w$ (such a c is called **coloring** of G).

If $G \preccurlyeq H$ then $\chi(G) \leq \chi(H)$ and $\gamma_o(G) \geq \gamma_o(H)$. Thus if $\chi(G) < \chi(H)$ and $\gamma_o(G) < \gamma_o(H)$, then G and H are $\preccurlyeq$-incomparable.

Recall also that a graph G is **bipartite** iff $\chi(G) = 2$ iff $\gamma_o(G) = \infty$.

Example 1: controlling chromatic number, (odd) girth, etc...

Fix $1 \leq n \leq \aleph_0$ and $m, k \in \mathbb{N} \cup \{\infty\}$: we want to deal with the class $\mathcal{G}_{n,m,k}$ of graphs G with $\chi(G) = n$, $\gamma(G) = m$, and $\gamma_o(G) = k$.

Theorem (Erdős)

For every $3 \leq n \leq \aleph_0$ there are $G \in \mathcal{G}$ with $\chi(G) = n$ and arbitrarily high girth.

Thus, apart from two trivial limitations ($n = 2$ iff $k = \infty$; $m \leq k$) the class $\mathcal{G}_{n,m,k}$ is nonempty: when this happens, we call the triple (n, m, k) **acceptable**.

Questions

How many graphs are there in such classes? How complicated is their homomorphism structure $\preceq \restriction \mathcal{G}_{n,m,k}$? Can we classify elements of $\mathcal{G}_{n,m,k}$ up to $\approx$?

Example 1: controlling chromatic number, (odd) girth, etc...

A classical construction from category theory due to Pultr-Trnková provides a **categorical embedding** (= injective fully faithful functor) which can be interpreted as a Borel reduction from homomorphism on $\mathrm{Mod}_{\mathcal{L}}$ to $\preccurlyeq\restriction \mathcal{G}_{n,m,k}$, *under certain nontrivial constraints on* $\mathcal{L}$ *and* (n, m, k).

Proposition (Louveau-Rosendal + Pultr-Trnková)

Assume that either $n > 3$ is finite and $m = k = 3$, or $n = 3$ and $m = k > 3$ are arbitrary (but finite). Then $\preccurlyeq\restriction \mathcal{G}_{n,m,k}$ is invariantly universal.

Proof. Enlarge $\mathcal{L}_g$ to $\mathcal{L}$ by adding two binary relational symbols P and Q, and turn each graph $G \in \mathcal{G}$ into an $\mathcal{L}$-structure $G' \in \mathrm{Mod}_{\mathcal{L}}$ by interpreting P as the "non-edge" relation and Q as $\neq$. The map $G \mapsto G'$ is a Borel reduction from $\sqsubseteq$ to the homomorphism relation on $\mathrm{Mod}_{\mathcal{L}}$, which is also a categorical embedding. Compose it with the Pultr-Trnková embedding to get $F \colon \mathcal{G} \to \mathcal{G}_{n,m,k}$: then F simultaneously witnesses $\sqsubseteq \leq_B \preccurlyeq\restriction \mathcal{G}_{n,m,k}$ and $\cong \leq_B \cong\restriction \mathcal{G}_{n,m,k}$ and satisfies $\mathrm{Aut}(G) \cong \mathrm{Aut}(F(G))$ for every $G \in \mathcal{G}$ — it is known that these conditions suffice to ensure invariant universality. □

Example 1: controlling chromatic number, (odd) girth, etc...

For technical reasons, the Pultr-Trnková functor, which is based on the so-called "replacement operation" cannot be used to deal with the other acceptable triples (n, m, k).

With a completely different technique (**connected sums**) we provided more flexible categorical embeddings and get for example:

Theorem 1 (M.-Scamperti)

Let (n, m, k) be any acceptable triple. Then

- either $\mathcal{G}_{n,m,k}$ is a single $\approx$-class (if $n = 2$ or $n = m = k = 3$),
- or else $\preccurlyeq \restriction \mathcal{G}_{n,m,k}$ is invariantly universal (and hence complete for analytic quasi-orders).

Let's see how the new functor is constructed,
and how the proof of Theorem 1 is completed.

The functor: step 1

Colored graph (G, c): a graph G with a singled-out coloring c of G

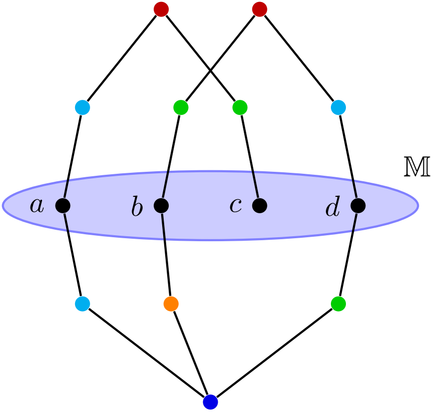
From $\mathcal{L}$-structures...

to colored graphs

$\mathcal{L} = \{P, Q\}$
$\mathrm{ar}(P) = 2$, $\mathrm{ar}(Q) = 3$

$\mathbb{M} = (M, P^{\mathbb{M}}, Q^{\mathbb{M}})$ with

$M = \{a, b, c, d\}$
$P^{\mathbb{M}} = \{(a, c), (d, b)\}$
$Q^{\mathbb{M}} = \{(a, d, b)\}$

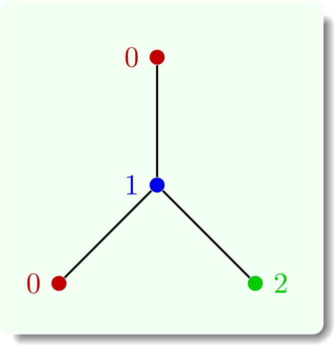


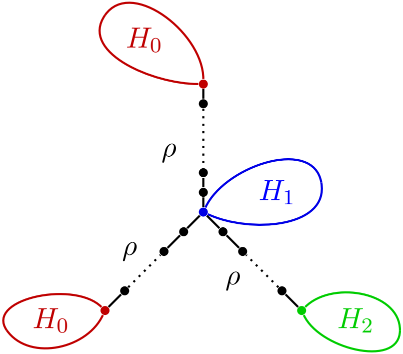
The functor: step 2

$\mathcal{H} = \{H_i \mid i \in \omega\}$ family of connected uniformly non-bipartite pairwise $\preccurlyeq$-incomparable rigid graphs of size $\leq \aleph_0$, and $\rho \in \mathbb{N}$ large enough.

From colored graphs (G, c)...

to the connected sum $\bigoplus_{\rho,c}^{G,\mathcal{H}} G_v$





$G_v \cong H_{c(v)}$ for every $v \in G$

Proof of Theorem 1

Key fact

If ρ is large enough, then for every $j \in \omega$ and every homomorphism $h\colon H_j \to \bigoplus_{\rho,c}^{G,\mathcal{H}} G_v$ there is a unique $\bar{v} \in G$ such that $h(H_j) = G_{\bar{v}}$, and moreover $c(\bar{v}) = j$ and h is the canonical isomorphism between H_j and $G_{\bar{v}}$.

[We can e.g. require $\rho \geq \min\{n_{H_i} - 2, \operatorname{diam}(H_i)\}$ for all $i \in \omega$.]

Once we have this, the proof boils down to finding a family $\mathcal{H}$ ensuring that $\bigoplus_{\rho,c}^{G,\mathcal{H}} G_v$ has the desired features. In the present case:

Lemma

Let $G = \bigoplus_{\rho,c}^{G,\mathcal{H}} G_v$ with ρ large enough, and let $I = \operatorname{rng} c$. Then

$$\chi(G) = \sup_{i \in I} \chi(H_i) \qquad \gamma(G) = \min_{i \in I} \gamma(H_i) \qquad \gamma_o(G) = \min_{i \in I} \gamma_o(H_i).$$

So it is enough to find a suitable family $\mathcal{H} \subseteq \mathcal{G}_{n,m,k}$.

Proof of Theorem 1

Theorem (M.-Scamperti)

Let (n, m, k) be acceptable with $n \geq 3$ and one of n, m different from 3. Then there is a family $\mathcal{H}$ as before such that

- $\operatorname{diam}(\mathcal{H}) = \sup_{i \in \omega} \operatorname{diam}(H_i) < \aleph_0$
- H_i is finite if so is n, and $|H_i| = \aleph_0$ otherwise
- $H_i \in \mathcal{G}_{n,m,k}$.

Together with the tricks mentioned before,
this concludes the proof of Theorem 1.

Remark

The chromatic number can be replaced by the **circular chromatic number** χ_c, the **fractional chromatic number** χ_f, and so on.

Example 2: Forbidden graphs

In graph theory (and its applications), a prominent role is played by classes of graphs omitting certain configurations. More precisely, given a collection of connected graphs $\mathcal{F}$ we look at

$$\mathrm{Forb}_{\mathcal{F}} = \{G \in \mathcal{G} \mid F \not\preccurlyeq G \text{ for all } F \in \mathcal{F}\}.$$

The class $\mathrm{Forb}_{\mathcal{F}}$ is $\preccurlyeq$-downward closed: thus it is closed under products $\times$, and obviously it is also closed under sums $\oplus$, i.e. it is an **ideal class**.

One of the best known results concerning the structure of $\mathrm{Forb}_{\mathcal{F}}$ was:

Theorem (Nešetřil-Rödl)

Every $\mathrm{Forb}_{\mathcal{F}}$, if not trivial, contains an infinite set of $\preccurlyeq$-incomparable graphs.

Example 2: Forbidden graphs

Notice that if the one-point graph K_1 belongs to $\mathcal{F}$, then $\mathrm{Forb}_{\mathcal{F}} = \emptyset$.

Theorem 2 (M.-Scamperti)

Let $\mathcal{F}$ be a collection of connected graphs not containing K_1. Then exactly one of the following alternatives holds:

1. $\mathrm{Forb}_{\mathcal{F}}$ consists of the discrete graph.
2. $\mathrm{Forb}_{\mathcal{F}}$ consists of all bipartite graphs.
3. The homomorphism relation on $\mathrm{Forb}_{\mathcal{F}}$ is invariantly universal.

Proof (sketch). We can assume that $\mathcal{F}$ is $\preccurlyeq$-upward closed. If $\mathcal{F}$ contains a bipartite graph we are in case 1, while if $\mathcal{F}$ contains all odd circular graphs C_j we are in case 2. In all remaining cases, there is an odd $j \geq 3$ such that $C_j \in \mathrm{Forb}_{\mathcal{F}}$. Then we can construct a family $\mathcal{H}$ such that $\gamma(H_i) = \gamma_o(H_i) = j+2$ and $H_i \preccurlyeq C_j$ for all $j \in \omega$, so that each connected sum $G = \bigoplus_{\rho,c}^{G,\mathcal{H}} G_v$ given by our functor satisfies $G \preccurlyeq C_j$ when ρ is even. Then $G \in \mathrm{Forb}_{\mathcal{F}}$ and we are in case 3. □

An empirical remark

In all applications, our method reveals a sort of general dichotomy, which can be stated in two forms depending on whether we consider "invariant universality" (which requires rigidity of the graphs in $\mathcal{H}$), or just "completeness" (which can be obtained even without rigidity).

Suppose that $\mathcal{C}$ is closed under (sufficiently large) connected sums and restrictions to connected components. Assume further that, up to homomorphic equivalence, all graphs in $\mathcal{C}$ are (uniformly) non-bipartite.

1. Either $\preccurlyeq \restriction \mathcal{C}$ is *almost linear* (= all $\preccurlyeq$-antichains have size ≤ 2), or else $\preccurlyeq \restriction \mathcal{C}$ is complete for analytic quasi-orders.
2. If there are three *rigid* $\preccurlyeq$-incomparable graphs in $\mathcal{C}$, then $\preccurlyeq \restriction \mathcal{C}$ is even invariantly universal.

Planar graphs (work in progress)

A graph is **planar** if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that no edges cross each other. For example, K_4 is planar while K_5 is not.

In general, planar graphs are considered simpler than general graphs, especially when their vertices all have low degree. Surprisingly, Hubička and Nešetřil proved that this is not quite true.

Theorem (Hubička-Nešetřil, 2005)

Any countable partial order can be embedded into the homomorphism structure of *finite* cubic (= degree at most 3) planar graphs.

Moving to our framework, one can then ask how much complex is the homomorphism relation $\preccurlyeq$ on *countable* cubic planar graphs, and how difficult is to classify them up to $\approx$.

Planar graphs (work in progress)

A relation R on a Polish space X is **σ-compact** if it can be written as a countable union of compact subsets of X^2. For example, the homomorphism relation on the Polish space of cubic graphs is σ-compact.

Theorem (M.)

The homomorphism relation on (countable) cubic planar graphs is complete for σ-compact quasi-orders.

The proof builds on another result of Louveau-Rosendal and uses one of the several variations of the previous method. We also get a form of invariant universality:

Corollary (M.)

For every *σ-compact* quasi-order R there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C} \subseteq \mathcal{G}$ consisting of cubic planar graphs such that $R \sim_B \preccurlyeq\restriction \mathcal{C}$.

Planar graphs (work in progress)

This solves our first problem, as we precisely computed the Borel complexity of $\preccurlyeq$ on the given class.

As for the associated classification problem, since the equivalence relation E_1 is σ-compact one has $E_1 \leq_B \approx$, and thus we easily get the following strong anti-classification result:

Corollary (M.)

Cubic planar graphs cannot be classified up to $\approx$ using as complete invariants countable structures (up to isomorphism) or, more generally, orbits of a continuous Polish group action.

The same applies to planar graphs whose vertices have degree at most d, for any finite $d \geq 3$.

What if we remove the restriction on the degrees of the vertices?

Planar graphs (work in progress)

Theorem (M.)

The relation $\preccurlyeq$ on planar graphs (with no bound on the degree of their vertices) is complete for analytic quasi-orders.

Still checking if it is also invariantly universal, but I guess it is...

A project

- Study more classes of graphs naturally appearing in combinatorics. (Suggestions?)
- Better understand "uniform" properties of graphs, e.g. uniform non-bipartiteness.
- Is it "functorial" Borel reducibility useful elsewhere?

Thank you for your attention!