Conjugacy relation of Cantor minimal systems

F. García-Ramos Jagiellonian University & UASLP

joint work in progress with Deka, Kunde, Kasprzak and Kwietniak - (Deka et al)

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- This makes $Homeo(X)$ a Polish space.

Topological conjugacy

- Two TDSs (X_1, T_1) and (X_2, T_2) are **conjugated** if there exists a homeomorphism $f : X_1 \to X_2$ such that $f \circ T_1 = T_2 \circ f$.

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- Let

$$\mathcal{R}_{\approx}(X) = \{(T_1, T_2) : (X, T_1) \approx (X, T_2)\} \subset Homeo(X) \times Homeo(X)$$

the equivalence relation generated by conjugacy.

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- In particular this implies that $\mathcal{R}_{\approx}(K)$ is a complete analytic set.

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- We say R **is reducible to** R' **as a set (**$R \preceq_B R'$) if there exists a Borel function $f : P \times P \to P' \times P'$ such that $(x, y) \in R$ if and only if $f(x, y) \in R'$.

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- Why minimal systems?

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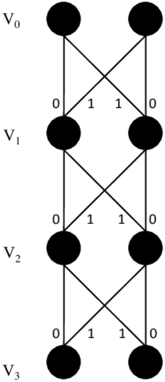
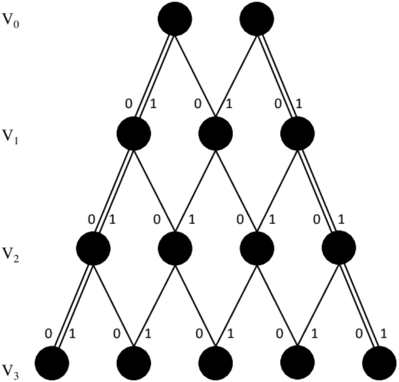
- For example Cantor minimal systems can be represented by transformations on Bratteli diagrams.





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- **Theorem** (Kaya '15) The equivalence relation generated by conjugacy of pointed Cantor minimal systems is bi-reducible to $=^+$.

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- For $\{x_n\}, \{y_n\} \in \mathbb{R}^{\mathbb{N}}$, we write $\{x_n\} =^+ \{y_n\}$ if $\{x_n : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\}$.

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- Given a TDS we define the topological Koopman operator $U_T : C(X) \to C(X)$ as $U_T(f) = f \circ T$.
- **Theorem** (Halmos - von Neumann) Two minimal equicontinuous systems are conjugated if and only if the eigenvalues of topological Koopman operator are the same.

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- This result uses the measurable version of the Halmos - von Neumann the eigenvalues of the operator on $L^2(X, \mu)$.

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- Before mentioning some tools...

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- **Theorem (**Foreman, Rudolph, Weiss '11) The isomorphism equivalence relation for ergodic measure preserving transformations is complete analytic.

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- isomorphic to the unique invariant measure of the subshift which is the reverse of $f(t)$ (using σ^{-1}).
- One uses the fact that the collection of ill-founded trees is a complete analytic set.

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- If one was able to "fix" this then we would conclude that the conjugacy relation for subshifts is not Borel.
- Actually the conjugacy between any subshifts is given by (finite-window) sliding-blockcodes, so the relation of conjugacy of subshifts is countable and hence Borel. Hence, this approach is impossible

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- We construct the Cantor subshifts step by step by enumerating the tree.
- At each step n we set the language of length $l(n)$ of the first m levels of the Cantor subshift (where m is the depth of the vertex n).

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- Every Cantor system is conjugated to a Cantor subshift.
- Not every Cantor subshift is a Cantor system, but every Cantor subshift without isolated points is a Cantor system.

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- With our previous approach we cannot obtain the result for flip conjugacy because a system is always flip-conjugated to its inverse.

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- $t \in Trees$ is ill founded if and only if $(f_2(t), \sigma)$ is conjugated to $(f_2(t), \sigma)$.
- **Theorem (Deka et al)** The flip conjugacy relation for Cantor minimal systems is complete analytic.

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      - the cyclic groups of prime order,
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    - one of 26 groups called the "sporadic groups"
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    - the Tits group (which is sometimes considered a 27th sporadic group).[note 1]
- **Theorem** (Robert '23) The relation obtained from isomorphisms of locally finite simple groups arises from a maximal S_∞-action.

Groups

- We define the **topological full group** of a TDS (X, T), $[[T]]$ as the subgroup of points $g \in Homeo(X)$ for which there exists a continuous function $f_g : X \to \mathbb{Z}$ such that $g(x) = T^{f_g(x)}(x)$.

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- Let (K, T) be a Cantor minimal system.

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- $[[T]]'$, the commutator of $[[T]]$ is simple (Matui '06, Bezuglyi-Medynets '07)

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- **Theorem (**Bezuglyi-Medynets '07) $(K, T_1) \approx_{flip} (K, T_2)$ if and only if $[[T_1]]'$ is isomorphic to $[[T_2]]'$.

Proposition

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- **Proposition** (Deka et al) The relation obtain by flip-cojugacy of Cantor minimal systems reduces to the relation of isomorphism of countable simple amenable groups.

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Open questions in descriptive set theory and dynamical systems

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- Dzieki!