Connes-Weiss and Glasner-Weiss theorems for Kazhdan equivalence relations, and applications to cost

Łukasz Grabowski
Leipzig University
j/w with Hector Jardon Sanchez and Sam Mellick

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Outline

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Theorem (Connes-Weiss)

G has property (T) if every free ergodic action of G is expanding.

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The group G does not have property (T) if for every free ergodic action π there exists a free ergodic extension σ such that σ has almost invariant sets.

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$$K(n) := \inf_{\pi} K_\pi(n)$$

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• G has property (T) iff for all n the Kazhdan constant $K(n)$ is non-zero. Furthermore, if G has property (T) then "infimum is relaised by some partition" i.e. for every n there exists an ergodic action π and an n-partition $\mathcal{A}$ such that $K(n) = K_\pi(n) = \mu(\partial\mathcal{A})$.

If an n-partition $\mathcal{A}$ is such that $K(n) = \mu(\partial\mathcal{A})$ then we say that $\mathcal{A}$ is *Kazhdan-optimal*.

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• Let $U \subset X$. We say that U has *almost unique clusters* if for almost all $x \in X$ the restriction of the graph $\mathcal{G}(x)$ to $U \cap G.x$ has finitely many infinite components.

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- This implies that the cost of a group with property (T) is 1.

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• Suppose BWOC that none of the parts have almost unique clusters. It is easy to see that we can find parts $\mathcal{A}$ and B such that $\mu(A) \geqslant \frac{1}{n}$, $\mu(B) \leqslant \frac{1}{n}$ and $\mu(S.A \cap B) \neq 0$, i.e. there are some edges between the A-clusters and B-clusters.

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• Consider the extension (Y, ν) which arises as Bernoulli on clusters of A, i.e. each cluster of A gets a number 0 or 1. Assume that probability of getting 0 is $\frac{1}{n^3}$.

• We define a partition of Y by first pulling back the partition $\mathcal{A}$, and then merging the A-clusters which got 0 with the B-clusters. The assumption that $\mathcal{A}$ doesn't have almost unique clusters implies that (after passing to an ergodic decomposition) we can just as well assume that Y is ergodic. This contradicts the Kazhdan-optimality of $\mathcal{A}$. □

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• The notions such as "ergodic", "expanding" and "almost invariant sets" apply to graphings just as well as they do to group actions.

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We say that a URG $\mathcal{G}$ has property (T), if every realisation of $\mathcal{G}$ is expanding.
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The key in the proof is to construct a Gaussian extension of a given graphing. Other approaches to Connes-Weiss might be possible, ask Francois Le Maître about full groups.

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• *A URG $\mathcal{G}$ has property (T) iff for all n the Kazhdan constant $K(n)$ is non-zero. Furthermore, if $\mathcal{G}$ has property (T) then "infimum is realised by some partition" i.e. for every n there exists a graphing (X, E, μ) and an n-partition $\mathcal{A}$ of X such that $K(n) = K_\pi(n) = \mu(\partial\mathcal{A})$.*

As before, if an n-partition $\mathcal{A}$ is such that $K(n) = \mu(\partial\mathcal{A})$ then we say that $\mathcal{A}$ is *Kazhdan-optimal*.

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Theorem (G-Jardon Sanchez-Mellick)

Let $\mathcal{G}$ be a URG with property (T). Suppose that (X, E, μ) is its realisation and and suppose that for some n we have a Kazhdan optimal n-partition $\mathcal{A}$ of X.

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Let $\mathcal{G}$ be a URG with property (T). Suppose that (X, E, μ) is its realisation and and suppose that for some n we have a Kazhdan optimal n-partition $\mathcal{A}$ of X. Then one of the parts of $\mathcal{A}$ has almost unique clusters.

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Let G be a locally-compact group, and let $\mathcal{E}$ be the equivalence relation associated to a Poisson point process on G. Then G has property (T) iff $\mathcal{E}$ has property (T).

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This leads also to examples of URG's with prperty (T) which don't arise from group actions in any obvious way.

Thank you

Thank you for your attention!