Continuous and open linings and treeings

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The main results presented are joint with Cody Olsen.

We present some results and questions concerning clopen or open structures on equivalence relations induced by free continuous actions of $\mathbb{Z}^n$.

We use a combination of techniques including forcing, hyperaperiodicity, and orthogonality arguments.

Statement of results

Let X be a Polish space, G a finitely generated marked groups, and $G \curvearrowright X$ a continuous free action of G on X. Let E be the corresponding equivalence relation.

Definition

A k-treeing of E is a subset T of the Schreier graph Γ such that on each class $[x]$, $T \restriction [x]$ is a vertex disjoint union of exactly k trees. A $\leq k$ treeing is where every $T \restriction [x]$ is a vertex disjoint union of $\leq$ trees.

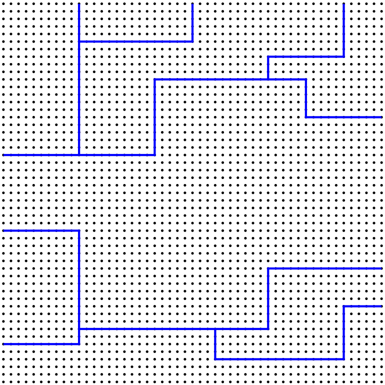


Figure: A $k = 2$ treeing

A special case is that of a k-lining:

Definition

A k-lining of E is a subset T of the Schreier graph Γ such that on each class $[x]$, $T \restriction [x]$ is a vertex disjoint union of exactly k lines (an acyclic graph with every vertex degree 2). We similarly define a $\leq k$ lining.

- If the domain T is all of X, we say T is a complete k-treeing or k-lining, etc.

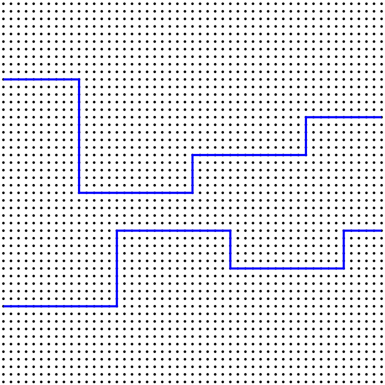


Figure: A $k = 2$ lining

We define the notions of the treeing or lining being Borel, clopen, or open in the natural way:

- We say a treeing or lining (T, E) is Borel if the set of edges is a Borel subset of $X \times X$. (It follows that T is Borel as well).
- We say the treeing or lining (T, E) is clopen (resp. open) if for each $g \in G$ $\{x : (x, g \cdot x) \in E\}$ is relatively clopen (resp. open) in $F(X)$.

Some Previous Results

Theorem (Marks-Unger, Gao-J-Krohne-Seward)

There is a Borel complete lining of $F(2^{\mathbb{Z}^n})$.

Theorem (Gao-J-Krohne-Seward)

There is no clopen lining on $F(2^{\mathbb{Z}^n})$ *for any* $n \geq 2$.

Theorem (Gao-J-Krohne-Seward, Grebik-Rozhon, Weilacher, Bencs-Hruskova-Tóth)

There is a Borel matching of $F(2^{\mathbb{Z}^n})$ *for any* $n \geq 2$.

We state some results which use a combination of forcing and hyperaperiodicity arguments.

Theorem

Let E be generated by the continuous free action of $\mathbb{Z}^n$ on a 0-dimensional space. Then E does not admit an open k-treeing for any $k \geq 1$.

Corollary

$F(2^{\mathbb{Z}^n})$ does not admit an open k-lining for any k and any $n \geq 2$.

On the other hand we have the following.

Theorem

Let E be generated by the continuous free action of $\mathbb{Z}^n$ on a 0-dimensional space. Then E has an open $\leq n+1$ treeing.

Recently we have improved this to:

Theorem

Let E be generated by the continuous free action of $\mathbb{Z}^2$ on a 0-dimensional space. Then E has an open ≤ 5 lining.

The following are still open:

Question

Does $F(2^{\mathbb{Z}^2})$ have a clopen $\leq k$ lining for some k?

Question

Does $F(2^{\mathbb{Z}^2})$ have a clopen $\leq k$ treeing for some k?

Hyperaperiodicity

Let $\Gamma \curvearrowright X$ be a continuous action of G on the Polish space X. Let E be the induced equivalence relation on X.

Definition

We say $x \in X$ is hyperaperiodic if $\overline{[x]} \subseteq F(X)$, the free part of the action.

We say $x \in 2^G$ is hyperaperiodic if it hyperaperiodic as an element of the (left) shift action of G on 2^G.

Theorem (Gao-J-Seward)

For every countable group G there is a hyperaperiodic element.

There is a combinatorial condition on $x \in 2^G$ equivalent to it being a hyperaperiodic element.

$$\forall s \neq 1_G \, \exists T \in G^{<\omega} \, \forall g \in G \, \exists t \in T \; x(gt) \neq x(gst)$$

For $G = \mathbb{Z}^n$ these elements are easy to construct directly.

There is a forcing notion $\mathbb{P}_{\mathrm{gp}}$, the grid-periodicity forcing which adjoins a hyperaperiodic element x_G of $F(2^{\mathbb{Z}^n})$ with extra properties (we use $n = 2$):

- x_G is a minimal element.
- For every k, $x \upharpoonright [-k, k]^n$ occurs with a period (m, m), for some m.

Grid periodicity forcing

Let $n \in \mathbb{Z}^+$. The grid periodicity forcing $\mathbb{P}_{\text{gp}}$ is defined as follows.

- A condition $p \in \mathbb{P}_{\text{gp}}$ is a function $p \colon R \setminus \{u\} \to \{0,1\}$ where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{Z}^2$ and $u \in R$. Also, both the width $b - a + 1$ and height $h = d - c + 1$ are powers of n.
- $q \leq p$ if R_q is obtained by a rectangular tiling by copies of R_p. If $c \in R_q$ is in the copy $R_p + t$ and $c - t \neq u_p$, then $q(c) = p(c - t)$. Also, u_q is one of the translated copies of u_p.

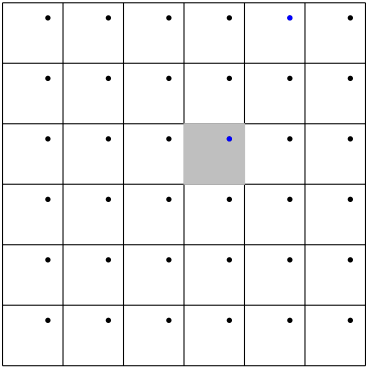


Figure: The extension relation in the grid periodicity forcing $\mathbb{P}_{gp}$.

Let $\mathbb{P} = \mathbb{P}_{\mathrm{gp}}$ and let x_G be generic for $\mathbb{P}_{\mathrm{gp}}$.

Lemma

x_G is hyperaperiodic and minimal.

Proof: Fix $s \neq 1_G = (0, 0)$. Let $p \in \mathbb{P}_{\mathrm{gp}}$. There is a $q \leq p$ such that $u_q + s \in \mathrm{dom}(q)$. There is an $r \leq q$ with two copies q_1, q_2 of q (except for $u_1 = u(q_1)$, $u_2 = u(q_2)$) and with $r(u_1) \neq r(u_2)$ (with both defined). Then $T = \mathrm{dom}(r)$ witnesses the statement of hyperaperiodicity for s. By density, x_G is hyperaperiodic.

The proof of minimality for x_G is similar to the next lemma.

Lemma

Let $A \subseteq \mathbb{Z}^2$ be finite. Then there is a lattice $L \subseteq \mathbb{Z}^2$ such that $x_G \restriction A = x_G \restriction (A + (a,b))$ for any $(a,b) \in L$.

Proof: Fix $A \subseteq \mathbb{Z}^2$ and $p \in \mathbb{P}_{\mathrm{gp}}$. There is a $q \leq p$ such that $A \subseteq \mathrm{dom}(q) \setminus u(q)$. If $\mathrm{dom}(q)$ has side lengths a, b, then can take $L = \mathbb{Z}(a,0) + \mathbb{Z}(0,b)$.

Nonexistence of open k-treeings

We sketch the proof of the following.

Theorem

For any $n \geq 2$ and any $k \geq 1$, there is no open k-treeing of $F(2^{\mathbb{Z}^n})$.

We take $n = 2$ for simplicity.

We let $\mathbb{P} = \mathbb{P}_{\text{gp}}$ be the grid-periodicity forcing for joining an element of $F(2^{\mathbb{Z}^2})$.

Let x_G be generic for $\mathbb{P}$.

- x_G is hyperaperiodic.
- x_G is also a minimal element.

Let $K = \overline{[x_G]}$. $K \subseteq X = F(2^{\mathbb{Z}^2})$ is compact.

Let $T_1, \dots, T_k$ be the trees on $[x_G]$.

Let $p \in \mathbb{P}$ be such that

$$p \Vdash \forall_{1 \leq i \leq k}\ g_i \cdot \dot{x}_G \in T$$
$$\wedge\ \forall_{i \neq j}\ g_i \cdot \dot{x}_G, g_j \cdot \dot{x}_G \text{ are not in the same } T \text{ component.}$$

Say $U \approx p \in 2^{[-N_0,N_0]^2}$ be the basic open set corresponding to p. Without loss of generality we may assume $\|g_i\| < N_0$.

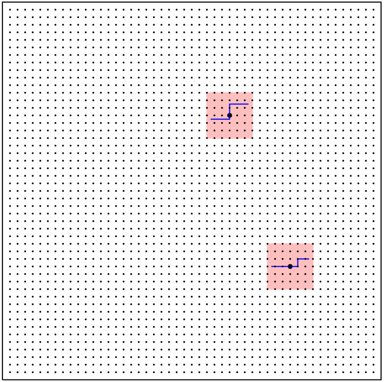


Figure: The generic class for $k = 2$

Since T is open, we may assume that for some $m < N_0$ that the $m \times m$ neighborhood V_i about each $g_i = (a_i, b_i)$ is contained in p and determines that $g_i \cdot x \in T$.

By grid periodicity, there is an $N_1 > N_0$ such that (N_1, N_1) is a period for U in x_G.

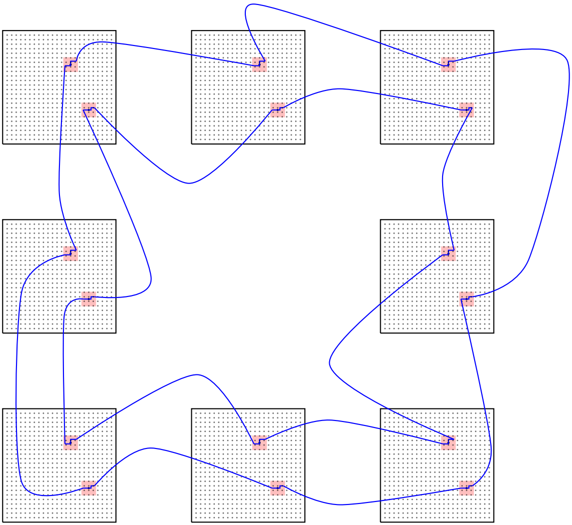
For each $x \in K$ and each set s of occurrences of $k + 1$ many neighborhoods $W_1, \dots, W_{k+1}$ in $x \restriction [-2N_1, 2N_1]^2$, where each W_i is one of the $V_1, \dots, V_k$, there is an n_x^s such that $x \restriction [-n_x^s, n_x^s]$ determines a path in a component of T between two of the center points of a W_i and a W_j, $i \neq j$.

By compactness of K, there is an $N_2 > N_1$ such that for all $x \in K$ and any occurrence s of $W_1, \dots, W_{k+1}$ in $x \restriction [-2N_1, 2N_1]^2$, two of the center points are connected by a path in T of length $< N_2$.

Consider now a rectangular "ring" of copies of U, with the spacing between adjacent copies N_1. This can be found in x_G by definition of N_1. The side length of the ring is at least $3N_2$.

Let $U^1, \dots, U^\ell$ denote these copies of U in x_G. Let $V_1^i, \dots, V_k^i$ denotes the corresponding copies of $V_1, \dots, V_k$ in U^i.

Let $x_1^i, \dots, x_k^i$ be the shifts of x_G which are centered at the copies of $V_1^i, \dots, V_k^i$ respectively.



For each $1 \leq i \leq \ell$, consider the points $x_1^i, \dots, x_k^i, x_1^{i+1}$.

Two of these points must be connected by a path of length $\leq N_2$ By genericity and the definition of P (and since shifting is an automorphism of $\mathbb{P}$), the path must connect x_1^{i+1} with one of the x_a^i.

Repeating the argument, we have that all of the x_a^i are connected to one of the x_b^{i+1} by a path of length $< N_2$.

These paths connect distinct points with distinct points.

This gives a set of k paths in T starting and ending at the $x_0^1, \dots, x_k^1$. One of these k paths must start and end at the same point x_i^1 (since U forced that distinct x_i^1 are not connected in T).

This gives a cycle in T.

This cycle is non-trivial as the side lengths of the ring are $> 3N_2$, and the paths from one U^i to U^{i+1} is at most N_2.

Existence of open ≤ 3 treeings for $F(2^{\mathbb{Z}^2})$

We sketch the proof of the following theorem.

Theorem

There is an open ≤ 3 treeing of $F(2^{\mathbb{Z}^2})$.

Let $d_0 < d_1 < \cdots$ be fast growing.

For each i, there is a clopen tiling $\mathcal{R}_i$ of $F(2^{\mathbb{Z}^2})$ by rectangles with side lengths in $\{d_i, d_i + 1\}$.

We define a sequence of clopen treeings $T_0 \subseteq T_1 \subseteq \cdots$.

- Each component of any T_i is finite.
- Each component of a T_i is contained within $d_0 + \cdots + d_{i-1}$ of a rectangular region $R \in \mathcal{R}_i$.

Assume T_{i-1} has been defined.

For each $R \in \mathcal{R}_i$, let $T_i(R)$ be the component trees of T_{i-1} for which R is the least rectangle in $\mathcal{R}_i$ intersecting it.

Clearly $\cup T_i(R) \subseteq B_\rho(R, d_0 + \cdots + d_{i-1})$.

Add the shortest path between two trees in $T_{i-1}(R)$. This doesn't add any cycles. Continue until the trees in $T_{i-1}(R)$ are connected into a single tree.

The resulting tree is a component of $T_i(R)$.

Let $T = \bigcup_i T_i = \bigcup_i \bigcup_{R \in \mathcal{R}_i} T_i(R)$.

Claim

Each E class has at most 3 components of T.

Proof.

Suppose $x_1, \dots, x_4$ are E-equivalent and in different T components. Let $d = \max \rho(x_i, x_y)$. Choose i with $d_i \gg d$. Then $B_\rho(x_1, 2d)$ can intersect at most 3 distinct $R \in \mathcal{R}_i$. So, two of the x_i are connected in T_{i+1}.

□

We sketch a proof of the following.

Theorem

There is an open $\leq k$ lining of $F(2^{\mathbb{Z}^2})$ for some k (can take $k = 5$).

We make use of the following lemma.

Lemma

There is a sequence of clopen rectangular tilings $\mathcal{R}_i$ with side lengths in $\{d_1, d_i + 1\}$ such that for each i and each R_{i+1} rectangle R, R can be divided into at most 3 subrectangles S such that the rectangles in $\mathcal{R}_i$ which intersect S are of the same size and form a rectangular gird.

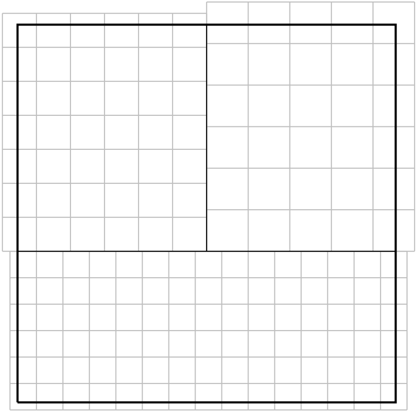


Figure: Statement of the Lemma

To do this, we use an auxiliary tiling $\tilde{\mathcal{R}}_{i+2}$ of scale d_{i+2}, and then subdivide each $\tilde{R} \in \tilde{\mathcal{R}}_{i+2}$ into $\approx d_i$ size subrectangles.

At stage i, we have three types of line segments: those following vertical boundaries of R_i rectangles (within $2d_{i-1}$), those following horizontal boundaries of R_i rectangles, and those which are internal to R_i rectangles.

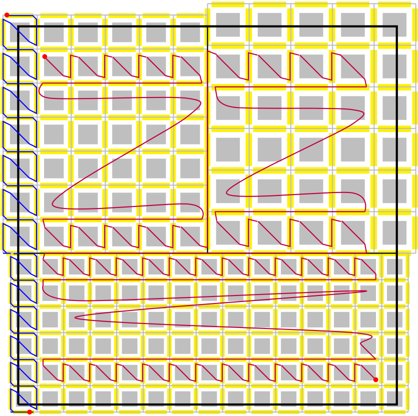


Figure: Inductive construction

Questions

Question

Does there exists a clopen $\leq k$ lining of $F(2^{\mathbb{Z}^n})$?

Question

What is the least k so that there is an open $\leq k$ treeing of $F(2^{\mathbb{Z}^n})$?

Question

Does there exists a clopen $\leq k$ treeing of $F(2^{\mathbb{Z}^n})$?