Generic analysis of Borel homomorphisms for the finite Friedman-Stanley jumps

Assaf Shani

Concordia University

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Borel homomorphisms and reductions

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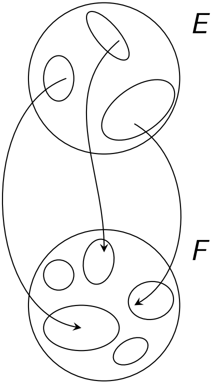

Definition

Let E and F be equivalence relations on Polish spaces X and Y respectively. $f\colon X \to Y$ a Borel map.

- f is a **Borel homomorphism**, $f\colon E \to_B F$, if $x \, E \, x' \implies f(x) \, F \, f(x')$.
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- E **is Borel reducible to** F, denoted $E \leq_B F$, if there is a Borel reduction of E to F.
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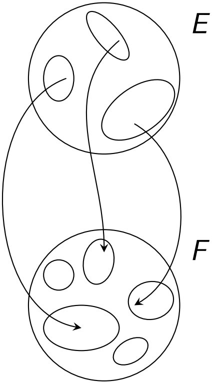

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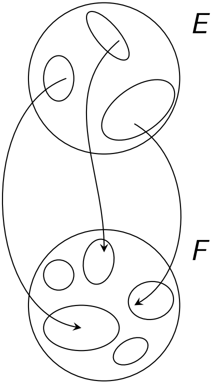

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- Possible complete invariants for classification problems.

Friedman-Stanley jump / countable powerset operation

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Let E be an equivalence relation on a Polish space X.
Define E^+ on the Polish space $X^{\mathbb{N}}$ by

$$x \; E^+ \; y \iff \forall n \exists m (x(n) \; E \; y(m)) \;\&\; \forall n \exists m (y(n) \; E \; x(m)),$$

that is, $\{[x(n)]_E ; \; n \in \mathbb{N}\} = \{[y(n)]_E ; \; n \in \mathbb{N}\}$.

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Remark:

For $=_{\mathbb{R}}^{+}$, the situation is well understood. Some examples:

- Foreman - Louveau 1995: $=_{\mathbb{R}}^{+}$ is Borel bireducible with the classification problem of ergodic discrete spectrum measure preserving transformations.
- Marker 2007: Let T be a first order theory whose space of types is uncountable. Then $=_{\mathbb{R}}^{+} \leq_B \cong_T$.

Generic dichotomy for Borel homomorphisms

Theorem (Kanovei-Sabok-Zapletal 2013)

Let E be an analytic equivalence relation. Then either

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- Can be done if A is a Scott set: sufficiently closed under some countably many operations.
- Improved idea: $A \mapsto \mathrm{closure}(A) \mapsto M$.
- This gives a Borel homomorphism, not trivial on comeager sets. Therefore $=^+_{\mathbb{R}} \leq_B \simeq_T$.

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More generally:

- For $n \geq 2$, need a different presentation / topology.
- Need to consider the homomorphisms $=^{+n}_{\mathbb{R}} \to_B =^{+k}_{\mathbb{R}}$, $k < n$, essentially taking a hereditarily countable set of rank n to the set of its rank k elements.

Main result

Theorem (S.)

There are equivalence relations F_n on Polish spaces X_n, s.t.

1. $F_n \sim_B =_{\mathbb{R}}^{+n}$, $n = 1, 2, 3, \ldots, \omega$, and

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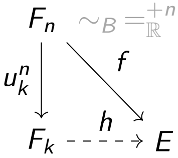

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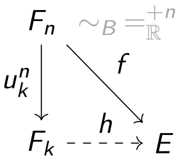
To prove that $=_{\mathbb{R}}^{+n} \leq_B E$, enough to find a non-trivial homomorphism.



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The following answers positively a question of Clemens.

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For any analytic equivalence relation E, either

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- From the definitions, F_ω is equivalent to its restriction to any fiber of u^ω_k.

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- From the definitions, F_ω is equivalent to its restriction to any fiber of u_k^ω.
- It remains to show that F_ω retains its complexity on comeager sets: $F_\omega \leq_B F_\omega \restriction C$ for any comeager C.

Spectrum of the meager ideal

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In particular, $=_{\mathbb{R}}^{+n}$ is in the **spectrum of the meager ideal**. This was proved by Kanovei, Sabok, and Zapletal for $n = 1$. *For $n > 1$, the theorem fails for $=_{\mathbb{R}}^{+n}$, so the F_n's are necessary.*

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$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
* & 1 & 0 & 1 & \ldots \\
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x(0) & x(1)(0) & x(1)(1) & x(1)(2) & \\
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- $A^x_2 = \left\{a^{x,l}_1;\ l \in \mathbb{N}\right\};\ a^{x,l}_2 = \left\{a^{x,k}_1;\ x(2)(l)(k) = 1\right\} \subseteq A^x_2$.

Definition of F_n and u_m^n

- $X_n \subseteq ((2^{\mathbb{N}})^{\mathbb{N}})^n$, for $n = 1, 2, 3, \dots, \omega$. Fix $x \in X_n$.
- $A_1^x = \{x(0)(k);\ k \in \mathbb{N}\} \subseteq 2^{\mathbb{N}}$.

$$a_1^{x,l} = \{x(0)(k);\ x(1)(l)(k) = 1\} \subseteq A_1^x$$

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
* & 1 & 0 & 1 & \dots \\
* & 1 & 1 & 0 & \dots \\
* & 0 & 1 & 1 & \dots \\
* & 0 & 1 & 0 & \dots \\
x(0) & x(1)(0) & x(1)(1) & x(1)(2) & \\
(2^{\mathbb{N}})^{\mathbb{N}} & 2^{\mathbb{N}} & 2^{\mathbb{N}} & 2^{\mathbb{N}} &
\end{array}
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\vdots & \vdots & \vdots & \\
* & - & * & \dots \\
* & * & - & \dots \\
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- $u^n_m : X_n \to X_m$, for $m < n$, projection.

What's good about F_n? Group action

$S_\infty = \mathrm{Sym}(\mathbb{N})$, $S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}} \rightsquigarrow =^+_{\mathbb{R}}$ (on a large set).

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$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & * & 1 & 0 & 1 & \ldots \\
 & * & 1 & 1 & 0 & \ldots \\
 & * & 0 & 1 & 1 & \ldots \\
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In contrast, $=^{++}_{\mathbb{R}}$ is naturally induced by an action of

$$S_\infty \ltimes (S_\infty)^{\mathbb{N}} \text{ on } (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$$

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Note: $=^+$ is $\mathbf{\Pi}^0_3$; $=^{++}$ is $\mathbf{\Pi}^0_5$; $=^{+++}$ is $\mathbf{\Pi}^0_7$.

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$=^{+n}$ is *potentially* $\mathbf{\Pi}^0_{n+2}$: it is Borel reducible to a $\mathbf{\Pi}^0_{n+2}$ ER.
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$$\forall n \exists m \left(a_1^{x,n} = a_1^{y,m} \right)$$

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$$\forall n \exists m (\forall i, j\, [\, x(0)(i) = y(0)(j) \rightarrow x(1)(n)(i) = y(1)(m)(j)\,])$$

*	1	1	0	*	0	1	1
*	0	1	1	*	0	0	0
*	0	1	0	*	1	1	0
$x(0)$	$x(1)(0)$	$x(1)(1)$	$x(1)(2)$	$y(0)$	$y(1)(0)$	$y(1)(1)$	$y(1)(2)$

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Fix map $g \colon 2^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ s.t. for $a \neq b \in 2^{\mathbb{N}}$, $g(a), g(b)$ are "sufficiently generic".

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Would want some $g\colon (2^{\mathbb{N}})^{\mathbb{N}} \times 2^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$, taking some set of reals A_1^x and some subset $a \subseteq A_1^x$, to infinitely many "very distinct" subsets of A_1^x.

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This cannot be done in a way which is independent of the enumeration of A_1^x.

Some ideas for $n \geq 2$

Small modification to $n = 1$ case: Fix $g_1 \colon (2^{\mathbb{N}})^{<\mathbb{N}} \to 2^{\mathbb{N}}$ s.t. for $a \neq b \in (2^{\mathbb{N}})^{<\mathbb{N}}$, $g(a), g(b)$ are "sufficiently generic". Define

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E.g.: given $\zeta, \xi \in 2^{\mathbb{N}}$, want the subsets corresponding to $g(\zeta), g(\xi)$ to be "very different". On the set on all $t \in \mathbb{N}^{<\mathbb{N}}$ for which $\zeta \circ t, \xi \circ t$ are different, the subsets behave like G.

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$$\begin{aligned} f_2 \colon (2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}} \to & (2^{\mathbb{N}})^{\mathbb{N}^{<\mathbb{N}}} \times (2^{\mathbb{N}^{<\mathbb{N}}})^{\mathbb{N}^{<\mathbb{N}}} \\ \sim & (2^{\mathbb{N}})^{\mathbb{N}} \quad \times (2^{\mathbb{N}})^{\mathbb{N}^{<\mathbb{N}}} \end{aligned}$$

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