Generic properties in the space of actions of non finitely generated groups on the Cantor space

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Joint work with M. Doucha and T. Tsankov

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A few conventions and definitions

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- *minimal* if all orbits are dense, equivalently there is no nontrival invariant closed subset.

The space of actions

The space $A(\Gamma)$ of actions of Γ on X carries a natural Polish topology. Basic open sets are of the form

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If α is a *factor* of β (i.e. there is a continuous Γ-equivariant map from (X, β) to (X, α)) then α belongs to the closure of the conjugacy class of β.

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- What can one say about generic properties of elements of $A(\Gamma)$?
- How do those generic properties depend on Γ?

Existence of comeagre conjugacy classes

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No Γ s.t. $A(\Gamma)$ has a comeagre conjugacy class and Γ is not finitely generated is known.

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Then $S_{\mathcal{A}}(X)$ is a subshift of n^Γ, and $S_{\mathcal{A}}$ is Γ-equivariant. That way one can see α as an inverse limit of subshifts.

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The space of subshifts on A carries the Vietoris topology; SFTs are dense so every isolated subshift is an SFT.

Doucha's criterion

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An element with a comeagre conjugacy class (when it exists) is an inverse limit of projectively isolated subshifts.

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I do not know any example of a minimal sofic subshift on a non-f.g. group.

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This question does admit a surprisingly neat answer for amenable groups.

Co-induction

Fix two countable groups $\Gamma \leq \Delta$, as well as an action of Γ on X. Δ naturally acts on X^Δ via $\delta_1 \cdot f(\delta_2) = f(\delta_1^{-1}\delta_2)$.

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Define $\pi\colon X^{\Delta} \to X$ by $\pi(f) = f(1)$.

Looking for a Δ-invariant $Y \subseteq X^{\Delta}$ such that π is Γ-equivariant, one ends up with

$$Y = \left\{ f \in X^{\Delta} \colon \forall \delta \in \Delta \ \forall \gamma \in \Gamma \quad \gamma \cdot f(\delta) = f(\delta\gamma^{-1}) \right\}$$

$\Delta \curvearrowright Y$ is called the *co-induced action*. Morally $Y = X^{\Delta/\Gamma}$ with a "twisted " action of Δ.

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Proof.
Pick U open nonempty in $A(\Delta)$. There are $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$, $\alpha\colon \Gamma \curvearrowright X$ and $\mathcal{A}$ a clopen partition of X s.t. any Δ-action β with $\beta(\gamma_i)_{|\mathcal{A}} = \alpha(\gamma_i)_{|\mathcal{A}}$ belongs to U.

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(A conjugate of) the co-induced action of $\alpha_{|\Gamma}$ belongs to U, and is topologically k-transitive for all k by the previous proposition. Conclude by observing that this is a G_δ condition. □

The amenable, non locally finite case

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$\alpha \in A(\Gamma)$ is *shrinking* if there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $U_1, \ldots, U_n \in \mathrm{Clopen}(X)$ s.t.
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Corollary
If Γ is infinite, amenable and not locally finite, a generic element of $A(\Gamma)$ is not minimal. Hence conjugacy classes in $A(\Gamma)$ are meagre if Γ is amenable and not finitely generated.

Definition
A Borel probability measure μ on X is a *good measure* if μ is atomless, has full support and for all $A, B \in \mathrm{Clopen}(X)$

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Theorem (Akin)
If μ, ν are good measures on X such that $V(\mu) = V(\nu)$ then $\exists g \in \mathrm{Homeo}(X)\ g_*\mu = \nu$.

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For a generic α, the unique α-invariant measure μ is good and

$$V(\mu) = \left\{ \frac{n}{|\Gamma|} \colon n \in \{0, \dots, |\Gamma|\},\ \Gamma \text{ a finite subgroup of } \Delta \right\}$$