Maximally highly proximal flows of locally compact groups

Todor Tsankov
(joint with Adrien Le Boudec)

Université Claude Bernard – Lyon 1

Descriptive Set Theory and Dynamics Conference
Warsaw, 21 August 2023

Some notions from topological dynamics

G–topological group. A G-flow is a continuous action $G \curvearrowright X$ on a compact, Hausdorff space X. A morphism $\pi \colon X \to Y$ between G-flows is a continuous, G-equivariant map. If π is surjective we say that Y is a factor of X, or that X is an extension of Y.

X is minimal if every orbit is dense.

A map $\pi \colon X \to Y$ is irreducible if every non-empty open set $U \subseteq X$ contains a fiber $\pi^{-1}(y)$. Equivalently for every proper closed $F \subsetneq X$, $\pi(F) \subsetneq Y$.

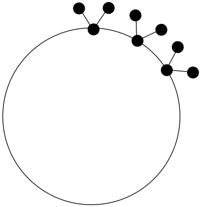
If X is metrizable, this is equivalent to π being almost one-to-one, i.e.,

$$\left\{x \in X : \pi^{-1}(\{\pi(x)\}) = \{x\}\right\} \text{ is dense } G_\delta.$$

We say that an extension $\pi \colon X \to Y$ of G-flows is highly proximal if π is irreducible.

Examples

Sturmyan subshifts



Start with an irrational rotation $x \mapsto x + \alpha$, choose one orbit and split every point in this orbit into two. There is a map from this (zero-dimensional) system to the circle which glues together the points we split. It is two-to-one on one (countable) orbit and one-to-one on the rest of the points.

Two circles

One can also split every point in the circle into two. The space becomes $S^1 \times \{0, 1\}$ with the lexicographic order and the order topology. This is also a highly proximal extension which is two-to-one everywhere and but it is not metrizable.

MHP flows

Every flow X admits a universal highly proximal extension $S(X) \to X$ with the following property: for every highly proximal extension $Y \to X$, there is a map $S_G(X) \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} S_G(X) & & \\ \downarrow & \searrow & \\ Y & \longrightarrow & X. \end{array}$$

A flow X is called maximally highly proximal (MHP) if it admits no proper highly proximal extensions; equivalently, if $S_G(X) = X$.

MHP flows were first considered by Auslander and Glasner for the minimal case and by Zucker in general.

MHP flows (cont.)

Highly proximal extensions preserve many dynamical properties:

- minimality
- proximality
- strong proximality
- etc.

This implies that the universal minimal flow, the universal minimal proximal flow, the universal minimal strongly proximal flow (the Furstenberg boundary) are all MHP.

The operation $X \mapsto S_G(X)$ is idempotent. Having a common HP extension is an equivalence relation on flows (given equivalently by $S_G(X) \cong S_G(Y)$) and MHP flows form a canonical transversal for it.

MHP flows are well-behaved in many situations. However, they are rarely metrizable: Zucker has proved that metrizability is equivalent to their isomorphic to the completion of a precompact homogeneous space G/H.

The Gleason cover

We denote by $\mathrm{BP}(X)$ the Boolean algebra of subsets of X with the Baire property and by $\mathrm{MGR}(X)$ the ideal of meager sets. We let $\hat{X}$ be the space of ultrafilters of the quotient algebra $\mathrm{BP}(X)/\mathrm{MGR}(X)$.

There is a natural map $\pi\colon \hat{X} \to X$ given by

$$\pi(p) \in U \iff U \in p \text{ for open } U \subseteq X.$$

An open subset $U \subseteq X$ is regular if $\mathrm{Int}(\overline{U}) = U$. Regular open sets form a canonical system of representatives for the quotient $\mathrm{BP}(X)/\mathrm{MGR}(X)$ and it is sometimes called the algebra of regular open sets.

The algebra $\mathrm{BP}(X)/\mathrm{MGR}(X)$ is complete and $\hat{X}$ is extremally disconnected: the closure of every open set is open.

The map π is irreducible and has the appropriate universal property with respect to irreducible maps.

Construction of the universal HP extension

If G is discrete, we can simply take $S_G(X) = \hat{X}$.

However, if G has non-trivial topology, the action $G \curvearrowright \hat{X}$ is, in general, not continuous. We would like to take the "continuous part" of the action.

Define:

$$\mathcal{B}(X) = \{f : X \to \mathbf{R} : f \text{ is Baire measurable and bounded}\}$$
$$\mathcal{M}(X) = \{f \in \mathcal{B}(X) : f = 0 \text{ on a comeager set}\}.$$

$\mathcal{B}(X)$ is a Riesz space: an ordered vector space with an archimedean unit $\mathbf{1}$, which is a lattice and $\mathcal{M}(X)$ is an ideal. $B(X) \coloneqq \mathcal{B}(X)/\mathcal{M}(X)$ is also a Riesz space with norm defined by

$$\|f\| > r \iff \{x \in X : |f(x)| > r \text{ is non-meager}\} \qquad \text{for } r \in \mathbf{R}.$$

We have that:

$$\hat{X} = \{p \in B(X)^* : p(f_1 \vee f_2) = p(f_1) \vee p(f_2), p(\mathbf{1}) = 1\}.$$

Construction of the universal HP extension (cont.)

E – Banach space, G–topological group $G \curvearrowright E$ by isometries. An element $f \in E$ is G-continuous if the map

$$G \to E, g \mapsto g \cdot f \text{ is continuous.}$$

The G-continuous elements form a closed subspace of E.

We let

$$B_G(X) = \{f \in B(X) : f \text{ is } G\text{-continuous}\}$$
$$\mathrm{S}_G(X) = \{p \in B_G(X)^* : p(f_1 \vee f_2) = p(f_1) \vee p(f_2), p(\mathbf{1}) = 1\}.$$

Then $G \curvearrowright \mathrm{S}_G(X)$ is a G-flow and we have HP maps:

$$\hat{X} \to \mathrm{S}_G(X) \to X.$$

The first construction of $\mathrm{S}_G(X)$ is due to Zucker and uses near ultrafilters on the algebra $\mathrm{BP}(X)/\mathrm{MGR}(X)$.

Functoriality properties

The construction above also tells us what are the correct functoriality properties of $S_G(\cdot)$.

A continuous map $\pi\colon X \to Y$ is category-preserving if $\pi^{-1}(F)$ is nowhere dense for every nowhere dense closed subset $F \subseteq Y$.

$S_G(\cdot)$ is a functor from the category of G-flows and category-preserving G-flow morphisms to the category of MHP G-flows.

For minimal flows, every G-flow morphism is automatically category-preserving.

The Chabauty topology

Y–locally compact, Hausdorff space; $F(Y) \coloneqq \{F \subseteq Y : F \text{ is closed}\}$. The Chabauty topology on $F(Y)$ is defined by a subbasis of sets of the form

$$\{F \in F(Y) : F \cap V \neq \varnothing\}, \quad V \subseteq Y \text{ open};$$
$$\{F \in F(Y) : F \cap K = \varnothing\}, \quad K \subseteq Y \text{ compact}.$$

The space $F(Y)$ is always compact, Hausdorff. If Y is discrete, $F(Y) = 2^Y$. If $Y = \bigcup K$ is represented as a directed union of compact subsets, then

$$F(Y) = \varprojlim F(K),$$

where each $F(K)$ is equipped with the Vietoris topology.

The space of subgroups and the stabilizer map

From now on, G is a locally compact group.

We define

$$\mathrm{Sub}(G) = \{H \in F(G) : H \text{ is a subgroup of } G\}.$$

$G \curvearrowright \mathrm{Sub}(G)$ by conjugation and it is a G-flow.

If $G \curvearrowright X$ is a dynamical system, we have a natural stabilizer map

$$\mathrm{Stab}\colon X \to \mathrm{Sub}(G), x \mapsto G_x := \{g \in G : x \in X\}.$$

Ideally, this map should allow to capture the information about stabilizers of the action in a convenient way. Works well for measure-preserving systems $G \curvearrowright (X, \mu)$: $\mathrm{Stab}_* \mu$ is an IRS.

The stabilizer map (cont.)

However, in the topological setting, the stabilizer map is usually not continuous:

$$\{x \in X : G_x \cap K = \varnothing\} \text{ is open for } K \subseteq G \text{ compact; but}$$
$$\{x \in X : G_x \cap V \neq \varnothing\} \text{ is in general not open for } V \subseteq G \text{ open.}$$

For discrete G, the second condition is equivalent to $\mathrm{Fix}(g) := \{x \in X : g \cdot x = x\}$ being open for every $g \in G$ (this fails, for example, for $\mathbf{Z} \curvearrowright 2^{\mathbf{Z}}$).

To overcome this difficulty, Glasner and Weiss suggested the following definition for minimal flows: the stabilizer URS (uniform recurrent subgroup) of the flow $G \curvearrowright X$ is defined as the unique minimal subflow of $\overline{\mathrm{Stab}(X)}$.

The main theorem

For discrete G, lower semicontinuity of the stabilizer map is equivalent to $\mathrm{Fix}(g) \coloneqq \{x \in X : g \cdot x = x\}$ being open for every $g \in G$.

Theorem (Frolík)

Let f be a homeomorphism of an extremally disconnected space. Then $\mathrm{Fix}(f)$ is open.

We prove a generalization of this theorem for locally compact groups.

Theorem

Let G be locally compact and let $G \curvearrowright X$ be an MHP flow. Then the stabilizer map $x \mapsto G_x$ is continuous.

Some consequences

In view of the theorem, we may associate to any flow X its stabilizer flow

$$\mathrm{Stab}(\mathrm{S}_G(X)) \subseteq \mathrm{Sub}(G).$$

This coincides with the stabilizer URS of Glasner and Weiss in the minimal case.

Denote by $\mathrm{Sa}(G)$ the greatest ambit (Samuel compactification) of G, the dual of the algebra of right-uniformly continuous bounded functions on G. $G \curvearrowright \mathrm{Sa}(G)$ is a G-flow.

Corollary (Veech)

Let G be locally compact. The action $G \curvearrowright \mathrm{Sa}(G)$ is free. In particular, G admits a free flow.

Proof.

The left action $G \curvearrowright G$ embeds densely in $\mathrm{Sa}(G)$ (as point evaluation), so $\{p \in \mathrm{Sa}(G) : G_p = \{1_G\}\}$ is dense. It is also closed by the theorem. $\square$

A word about the proof

Assume that G is second countable and let $\|\cdot\|$ be a proper norm on G (every closed ball is compact). Define a metric ∂ on X by:

$$\partial(x, y) = \inf\{\|g\| : g \cdot x = y\} \quad (\infty \text{ if in different orbits}).$$

If the flow is MHP, for every open set $U \subseteq X$, the function

$$x \mapsto \partial(x, \overline{U})$$

is continuous.

Main lemma

Let $g \in G$ and $r > 0$. Then there exist $n \geq 1$ and a continuous function $\phi: X \to \mathbf{R}^n$ such that for all $x \in X$

$$\partial(g \cdot x, x) > r \implies \|\phi(g \cdot x) - \phi(x)\|_\infty \geq r/3.$$