Some global aspects of transitive actions

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Based on joint works with Carderi, Fima, Gaboriau, Moon and Stalder

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We have two natural Polish models for the space of all transitive actions on infinite sets up to isomorphism.

First Polish model: the space of transitive actions on $\mathbb{N}$

Denote by $\mathrm{Hom}(\Gamma, S_\infty)$ the space of homorphisms $\Gamma \to S_\infty$, and by $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, S_\infty)$ the space of such homomorphisms which correspond to transitive actions.

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Let S_∞ act on $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, S_\infty)$ by $\sigma \cdot \alpha(\gamma) = \sigma\alpha(\gamma)\sigma^{-1}$. Then α is isomorphic to β iff they are in the same S_∞-orbit.

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Fact

$\mathrm{Sub}_{[\infty]}(\Gamma)$ *is* G_δ *in the Cantor space* $\{0,1\}^\Gamma$*, in particular it is Polish.*

Connection between the two spaces

Lemma (Glasner-Kitroser-Melleray, 2016)

The stabilizer map $\mathrm{Hom}_{\mathrm{tr}}(\Gamma, S_\infty) \to \mathrm{Sub}_{[\infty]}(\Gamma)$ *which takes* α *to* $\mathrm{Stab}_\alpha(0)$ *is open (as well as surjective and continuous).*

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Proposition

Let $\mathcal{P}$ *be an isomorphism invariant property of transitive* Γ*-actions, and let*

$$A_{\mathcal{P}} := \{\alpha \in \mathrm{Hom}_{\mathrm{tr}}(\Gamma, S_\infty) \colon \alpha \text{ has } \mathcal{P}\}$$
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Then $A_{\mathcal{P}}$ *is open/*G_δ*/dense iff* $B_{\mathcal{P}}$ *is.*

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- **amenable** if for every $\epsilon > 0$ and $S \Subset \Gamma$ there is an (S, ϵ)-invariant finite set, namely $F \Subset X$ such that for all $\gamma \in S$,

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Let $\Gamma = \Gamma_1 * \Gamma_2$ *with* $|\Gamma_1| \geq 2$ *and* $|\Gamma_2| \geq 3$. *Then the generic transitive action is highly transitive and totipotent, in particular faithful. Moreover, the generic transitive action is amenable iff* Γ *admits an amenable action all whose orbits are infinite.*

High transitivity

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- Every non-trivial normal subgroup of a highly transitive group is highly transitive, so solvable groups cannot be highly transitive as well.

More highly transitive groups

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Topological freeness on the boundary

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(1)⇒(2) is clear, (2)⇒(3) is an important result of Le Boudec and Matte Bon (2022), so what we really prove is (3)⇒(1).

A characterization of high transitivity for groups acting on trees

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1. *Γ is highly transitive;*
2. *Γ admits a faithful 4-transitive action;*
3. *The action $\Gamma \curvearrowright \mathcal{T}$ is topologically free on the boundary.*

4-transitive: for every partial bijection φ between finite subsets *of cardinality 4*, there is $\gamma \in \Gamma$ such that $\alpha(\gamma)$ extends φ.

(1)⇒(2) is clear, (2)⇒(3) is an important result of Le Boudec and Matte Bon (2022), so what we really prove is (3)⇒(1).

We use Baire category techniques, but in a space of actions satisfying additional restrictions. A key tool is Bass-Serre graphs of actions which we now present for free products.

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Lemma

This action is minimal of general type and topologically free on the boundary as soon as $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$.

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- Topological freeness allows us to find a large enough word $\gamma \in \Gamma_1 * \Gamma_2$ such that all the elements of $(F \sqcup \varphi(F))\alpha(\gamma)$ are in disjoint half-trees.
- One can then "connect" $F\gamma$ to $\varphi(F)\gamma$ using some $g \in \Gamma_1 \cup \Gamma_2$ so as to obtain $\tilde{\alpha}$ close to α such that $\tilde{\alpha}(\gamma g \gamma^{-1})$ extends φ. □

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Let $\Gamma \curvearrowright \mathcal{T}$ be a minimal irreducible faithful action. Suppose there are two edges $e_1, e_2 \in \mathcal{T}$ such that $\mathrm{Stab}(e_1) \cap \mathrm{Stab}(e_2)$ is finite.

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Let $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$. Then Γ admits a totipotent action.

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Corollary

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Azuelos and Gaboriau's result applies much more generally, yielding many groups with totipotent actions. They also have results for some hyperbolic groups.

Recall that an action is amenable if for every $\epsilon > 0$ and $S \Subset \Gamma$ there is an (S, ϵ)-invariant finite set, namely $F \Subset X$ such that for all $\gamma \in S$,

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Kazhdan groups have property (F), e.g. $\mathrm{Sl}_3(\mathbb{Z})$.

The Glasner-Monod characterization of $\mathcal{A}$ for free products

Theorem (Glasner-Monod, 2007)

A non-trivial free product $\Gamma_1 * \Gamma_2$ does not have $\mathcal{A}$ if and only if Γ_1 has (F) and Γ_2 has a finite index subgroup with (F), or vice versa.

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Totipotency simplifies greatly their proof: for the direct implication, it suffices to exhibit an amenable Γ-action all whose orbits are infinite (which Glasner and Monod do).

Two questions

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For $S_{(\infty)}$, answer is no, although it is highly transitive and has a totipotent action.