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THE SIMPLEX OF MEASURES INVARIANT UNDER DIFFEOMORPHISMS AND SABOK'S CONJECTURE

Matt Foreman
UC Irvine, August 14, 2023

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DON'T BURY THE LEDE

Theorem(Foreman, Weiss) Let K be compact metrizable Choquet simplex. Then there is a C^∞ map $T : \mathbb{T}^2 \to \mathbb{T}^2$ such that

- $M_T(\mathbb{T}^2)$ is affinely homeomorphic to K,
- T is Lebesgue measure preserving and ergodic.

THREE LECTURES

- Lecture 1: Background and motivation
- Lecture 2: Odometer based and circular systems: Global Structure Theorem
- Lecture 3: The simplex of invariant measures

HISTORICAL MOTIVATION

HAMILTONIAN SYSTEMS

Hamiltonian Systems were developed in the mid-19th century as a way of formalizing mechanical systems
(such as Newtonian Mechanics)

HAMILTONIAN SYSTEMS

A Hamiltonian system is described by a twice differentiable (C^2) function $H(\mathbf{q}, \mathbf{p})$ from $\mathbb{R}^{6N}$ to $\mathbb{R}$, giving the energy of the system. The system is described by *Hamilton's Equations*:

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}},$$
$$\frac{d\mathbf{q}}{dt} = +\frac{\partial H}{\partial \mathbf{p}}.$$

The variables $\mathbf{p}, \mathbf{q} \in \mathbb{R}^N$ are interpreted as the generalized position and momentum variables and the solution $r(t)$ is viewed as the trajectory of a point in an initial position $r(0) \in \mathbb{R}^{6N}$.

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H is the energy function

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LIOUVILLE'S THEOREM

Theorem The time trajectory $\{T_t\}_{t \in \mathbb{R}}$ preserves Lebesgue measure.

KHINCHIN'S THEOREM

Theorem The solutions to Hamilton's Equations with fixed energy

$$H = E$$

form a compact smooth manifold.

KHINCHIN'S THEOREM

Theorem The solutions to Hamilton's Equations with fixed energy

$$H = E$$

form a compact smooth manifold.

The smoothness of M matches the smoothness of H and the system restricted to M is ergodic.

ORIGINAL MOTIVATION:

STUDY THE PROBABILISTIC BEHAVIOR OF DIFFERENTIAL EQUATIONS AS THEY EVOLVE THROUGH TIME.

BACKGROUND

1. Objects: Measure preserving systems:

   $(X, \mathcal{B}, \mu, T)$

   Typically

   (a) $(X, \mathcal{B})$ is the Borel space of a Polish metric on X.

   (b) μ is a non-atomic complete separable probability measure on X.

   (c) $T : X \to X$ is an invertible measure preserving transformation sending elements of $\mathcal{B}$ to elements of $\mathcal{B}$.

BACKGROUND

1. Objects: Measure preserving systems:

   $(X, \mathcal{B}, \mu, T)$

2. What object is focussed on? The space? The transformation? The measure?

3. Do different spaces of measure preserving systems have substantially different properties?

4. Borel complexity questions

   (a) Are all interesting subsets Borel?

   (b) Do all subsets have the same complexity in different spaces?

   (c) How complicated is the isomorphism relation for different sets?

   (d) Does the answer depend on the setting?

BACKGROUND

1. Objects: Measure preserving systems:

   $(X, \mathcal{B}, \mu, T)$

2. What object is focussed on?

   (a) Is X a compact topological space? if so, how is that relevant? (Properties of T: homeomorphism? Borel map?)

   (b) Is X a manifold? if so, how is that relevant? (Properties of T: Diffeomorphism?)

   (c) Is X totally disconnected? Compact? (Properties of T: Is $X = \Sigma^{\mathbb{Z}}$ for some discrete set Σ? Is T the shift map?)

   (d) What are the properties of the unitary operator associated with T?

THE TOPOLOGY ON MPT

- If $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving transformation, then its *Koopman operator* is the map

$$U_T : L^2(X) \to L^2(X)$$

given by $U_T(f) = f \circ T^{-1}$.

- The Koopman operator is a unitary operator.
- Hence Koopman operators carry the *Weak Operator Topology*
- copying this over the the collection of measure preserving transformations on X gives Polish topology.
- We will refer to this space as MPT.

BACKGROUND

1. Objects: Measure preserving systems:

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3. Spaces of measure preserving systems

   (a) Do the measure preserving systems set in a given context form a nice space?

   (b) Do different spaces have different generic (dense $\mathcal{G}_\delta$) subsets?

   For example:

   (c) If they be presented as interval exchanges do they have the same generic properties?

   (d) Can they be presented so that they all have the same orbits (almost everywhere)?

   (e) Can every measure preserving transformation be presented as a diffeomorphism of a manifold?

TYPES OF QUESTIONS

- Questions about *settings*.
  In particular *realization problems*.

- Questions about Complexity

MOST PROMINENT REALIZATION PROBLEM

Is every measure preserving transformation isomorphic to a measure preserving diffeomorphism of a compact manifold?

OPEN EVEN IN A SPECIAL CASE

- Can a measure preserving diffeomorphism of a manifold be isomorphic to an odometer transformation?
- Can it be isomorphic to an odometer?

AN EASY OPEN QUESTION

Standard 0-1 laws for category show that

$\{T \in MPT : T \text{ can be realized by a diffeomorphism of a compact manifold}\}$

is either generic or its complement is generic.

Which is it?

AN EASY OPEN QUESTION

Standard 0-1 laws for category show that

$\{T \in MPT : T$ can be realized by a diffeomorphism of a compact manifold$\}$

is either generic or its complement is generic.

Which is it?

One answer wins the jackpot.

I conjecture the other direction—that a generic MPT *CAN* be realized.

MOST STANDARD SETTING

- $X = [0, 1]$
- μ is Lebesgue measure
- $\mathcal{B}$ is the completion of the Borel sets.

SYMBOLIC SHIFTS

- Let Σ be a finite or countable set.
- Let $\Sigma^{\mathbb{Z}} = \{f | f : \mathbb{Z} \to \Sigma\}$ be the $\mathbb{Z}$ product with the product topology.
- Let sh be the *shift map*:

$$sh(f)(n) = f(n+1)$$

TRANSLATIONS ON COMPACT GROUPS

Let G be a compact group and H be Haar measure. For each $g \in G$ define $T_g(h) = gh$. Then

$$(G, \mathcal{B}, H, T_g)$$

is a measure preserving system.

THE SPACE OF INVARIANT MEASURES

Let X be a Polish space and $T : X \to X$ be a Borel map. Then $M_T(X)$ is the space of T-invariant measures.

LONG HISTORY

- Bogoliubov-Krylov (1937) $M_T(X) \neq \emptyset$
- Banach-Alaoglu 1932 $M_T(X)$ is compact

ERGODIC TRANSFORMATIONS

A measure preserving system $(X, \mathcal{B}, \mu, T)$ is *ergodic* if

whenever $A \subseteq X$ is a T-invariant set either

- $\mu(A) = 0$ or
- $\mu(A) = 1$.

THE SPACE OF MEASURES

Let X be a compact Polish space, $\mathcal{B}$ be the Borel subsets of X and $T : X \to X$ be a Borel map. Then

- $\{\mu : \mu$ is a standard T-invariant measure on $\mathcal{B}\}$ is a metrizable Choquet simplex (with the weak* topology)
- The collection of ergodic measures are the extreme points of this simplex.

In particular, every invariant measure can be represented as an integral over the space of ergodic measures.

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We will often implicitly assume that the underlying space is compact.

ERGODIC DECOMPOSITION THEOREM

Consider $(X, \mathcal{B}, T)$. The fact $M_T(X)$ is a Choquet simplex and that the ergodic measures are the extreme points is a way of stating the

Ergodic Decomposition Theorem Let $\mu \in M_T(X)$ and $\mathcal{E}$ be the collection of ergodic measures (with the induced topology). Then there is a measure ν on $\mathcal{E}$ such that for every set $A \in \mathcal{B}$:

$$\mu(A) = \int \mu_i(A) d\nu(i).$$

WHAT SIMPLEXES OF INVARIANT MEASURES ARE POSSIBLE?

KRIEGER'S THEOREM

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system. Then there is a finite or countable set Σ and a shift invariant measure μ on $\Sigma^{\mathbb{Z}}$ such that

- $(\Sigma^{\mathbb{Z}}, \mathcal{C}, \mu, sh) \cong (X, \mathcal{B}, \mu, T)$
- $(\Sigma^{\mathbb{Z}}, \mathcal{C}, \mu, sh)$ is uniquely ergodic.

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- $(\Sigma^{\mathbb{Z}}, \mathcal{C}, \mu, sh)$ is uniquely ergodic.

So every ergodic transformation has a uniquely ergodic realization.

OXTOBY'S THEOREM

Theorem(Oxtoby 1952). There is a topologically minimal system (X, τ, T) that has exactly two ergodic transformations. Hence the simplex of invariant measures is a line.

WILLIAMS' THEOREM

Theorem(Williams 1984) If κ is finite, countable or cardinality $\mathbb{R}$, there is a topologically minimal system (X, τ, T) that has exactly κ ergodic transformations.

DOWNAROWICZ' THEOREM

Theorem(Downarowicz 1991) Let K be compact metrizable Choquet simplex. Then there is a topologically minimal systems such that the space of invariant measures is affinely homeomorphic to K.

Both Williams and Downarowicz Theorems use *Toeplitz Systems*.

TOEPLITZ SYSTEMS

Let Σ be finite, $\eta \in \Sigma^{\mathbb{Z}}$. Then

- $Per_n(\eta) = \{j \in \mathbb{Z} : \eta(j) = \eta(k) \text{ whenever } j = k(mod\ n)\}$
- $Aper(\eta) = \mathbb{Z} - \bigcup_n Per_n(\eta)$
- η is *Toeplitz* if $Aper(\eta) = \emptyset$.
- η is *dyadic Toeplitz* if $\mathbb{Z} = \bigcup_n Per_{2^n}(\eta)$.

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The orbit closure of a Toeplitz sequence is minimal.

Downarowicz construction used dyadic Toeplitz sequences

WHAT ABOUT MANIFOLDS?

Irrational rotations of the unit circle are uniquely ergodic.

So there are examples of diffeomorphisms
that are uniquely ergodic.

WHAT ABOUT MANIFOLDS?

Consider the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

- It has determinant 1 and determines a diffeomorphism of the torus.
- The simplex of invariant measures is a Poulsen Simplex.

So the collection of extreme points is dense

MAIN THEOREM

Theorem(Foreman, Weiss) Let K be compact metrizable Choquet simplex. Then there is a C^∞ map $T : \mathbb{T}^2 \to \mathbb{T}^2$ such that

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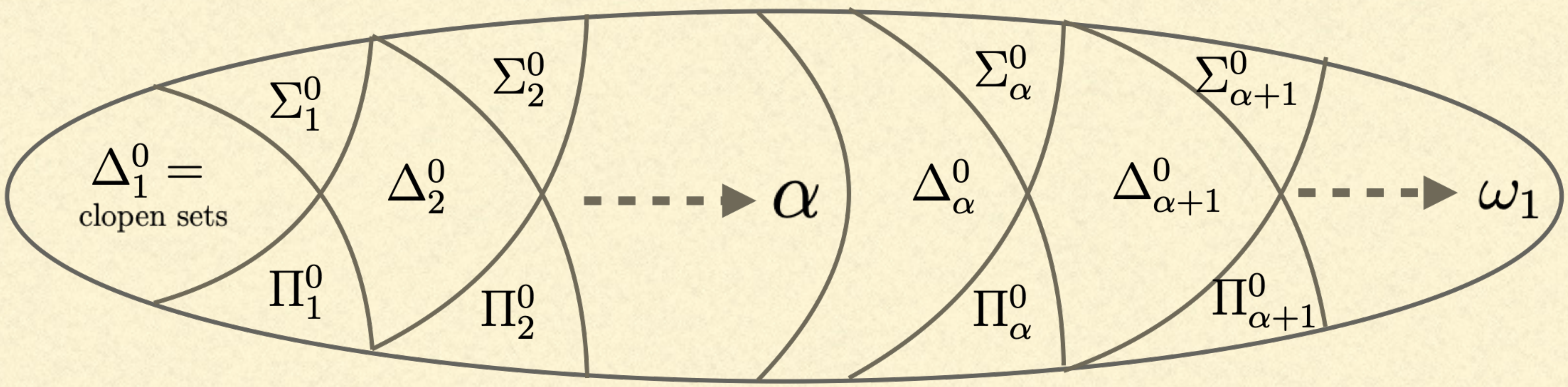
A REALIZABILITY THEOREM (of sorts)

DEFINABILITY

A PICTURE OF THE BOREL SETS



Let X be a Polish space

- A set $A \subseteq X$ is *analytic* if it is the continuous image of an open subset of some Polish space Y.
- A set $C \subseteq X$ is *co-analytic* if $X \setminus C$ is analytic.

FAMOUS MISTAKE OF LEBESGUE

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Lebesgue claimed in a paper in 1905 that every analytic set is Borel. This was corrected by Suslin in a paper published in 1917, where he gave a counterexample.

EXAMPLE: TREES

Let X be the space of countable, connected, acyclical, pointed graphs. Let A be the set of graphs with non-trivial ends. Then A is an analytic, non-Borel subset of X.

In a different context, X is called the space of *Trees* and A is called the set of ill-founded trees.

COMPARING SETS

The complexity of sets and equivalence relations are measured by *Reductions.* A set B is at least as complex as A if every questions about A can be reduced by a Borel function to a question about B.

REDUCTIONS: ONE DIMENSIONAL

Let X, Y be Polish spaces and $A \subseteq X, B \subseteq Y$.

- Then A is Borel reducible to B if and only if there is a Borel function $f : X \to Y$ such that
$$x \in A \text{ iff } f(x) \in B.$$

We write $A \preceq_{\mathcal{B}} B$.

Note that this is a transitive pre-ordering

REDUCTIONS: TWO DIMENSIONAL

Let X, Y be Polish spaces and $E \subseteq X \times X$, $F \subseteq Y \times Y$ be equivalence relations.

- Then E is Borel reducible to F if and only if there is a Borel function $f : X \to Y$ such that for all $x_1, x_2 \in X$

$$(x_1, x_2) \in E \text{ iff } (f(x_1), f(x_2)) \in F.$$

We write $E \preceq_{\mathcal{B}} F$.

Note that this is again a transitive pre-ordering

HEURISTIC

Let X, Y be Polish, $A \subseteq X, B \subseteq Y$ and E, F be equivalence relations on X and Y respectively.

- If $A \preceq_{\mathcal{B}} B$ then every question about membership in A can be reduced to a question about membership in B, so B is at least as complicated as A.
- If $E \preceq_{\mathcal{B}} F$ then any question about x_1, x_2 being E equivalent can by answered by a question about F equivalence. So the F equivalence classes are complete invariants for the relation E.

HEURISTICS

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In practice these heuristics are true.

If A is not Borel, then B is not Borel
If E does not have complete invariants then F doesn't either

COMMON SOURCES OF EQUIVALENCE RELATIONS

- Any calculable quantity or element of a Polish space gives an equivalence relation. (e.g. having the same Entropy)
- Polish Group actions. Being in the same orbit of a group is an equivalence relation

AN IMPORTANT EXAMPLE

Let S_∞ be the group of permutations of $\mathbb{N}$. Then S_∞ actions arise in the context of classification by countable structures such as countable abelian groups, isomorphism of countable graphs and many other contexts.

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These two examples are convenient because they are canonical examples of equivalence relations that are NOT Borel

SOME BENCHMARKS

- $=$
- E_0–the equivalence relation of eventual equality on $\{0,1\}^{\mathbb{N}}$,
- For X a Polish space E a given equivalence relation and $\vec{x}, \vec{y} \in X^{\mathbb{N}}$, $\vec{x} E^+ \vec{y}$ if there is a permutation ϕ of $\mathbb{N}$ such that $[x_n]_E = [y_\phi(n)]_E$.
- Isomorphism of countable graphs.

BENCHMARKS

- $=$ Complete numerical invariants
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BENCHMARKS

- $=$
- E_0–the equivalence relation of eventual equality on $\{0,1\}^{\mathbb{N}}$,

Equivalent to not having complete numerical invariants

- For X a Polish space E a given equivalence relation and $\vec{x}, \vec{y} \in X^{\mathbb{N}}$, $\vec{x} E^+ \vec{y}$ if there is a permutation ϕ of $\mathbb{N}$ such that $[x_n]_E = [y_\phi(n)]_E$.
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  Friedman Stanley jump
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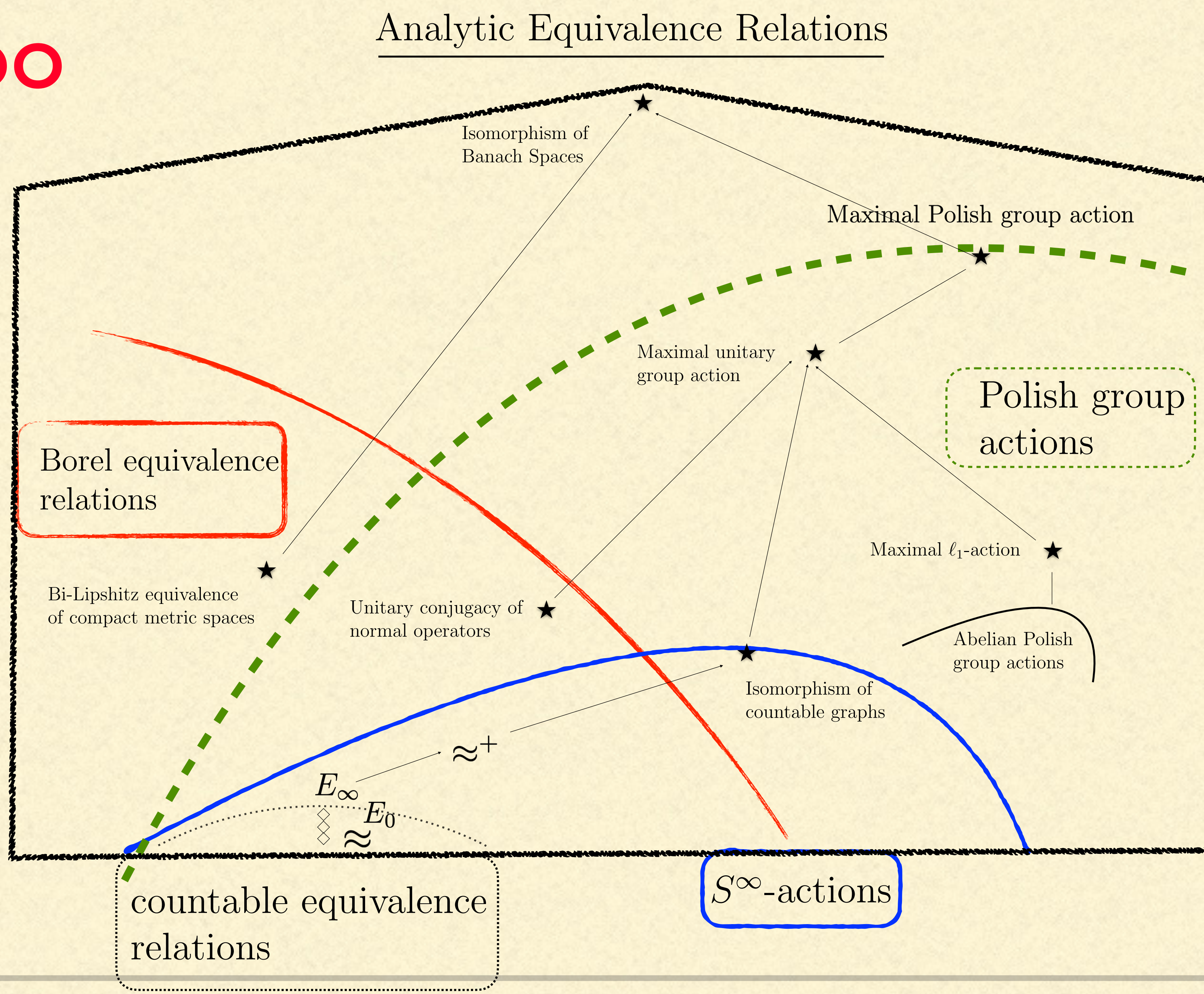
Maximal S-infty action

The Zoo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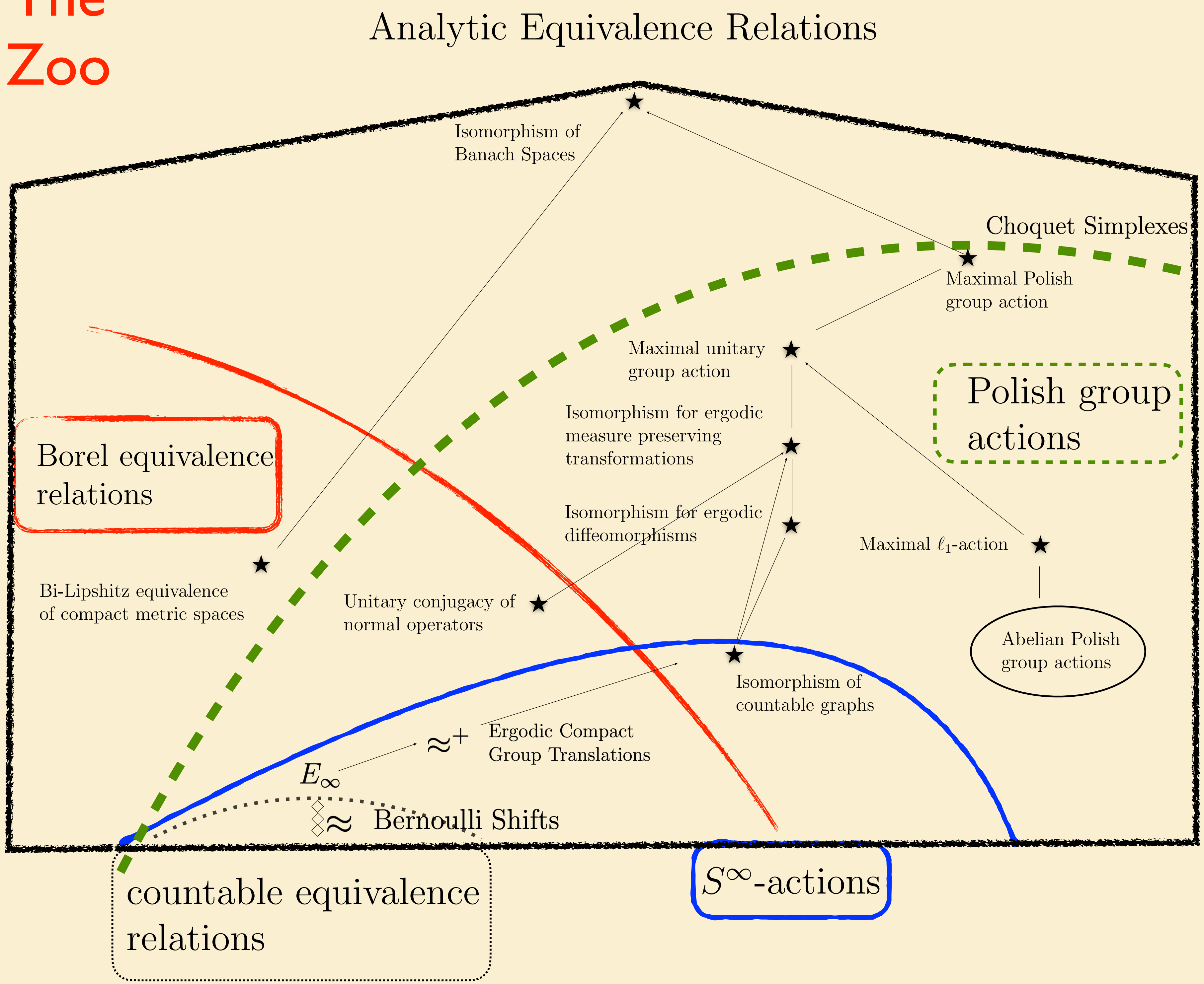
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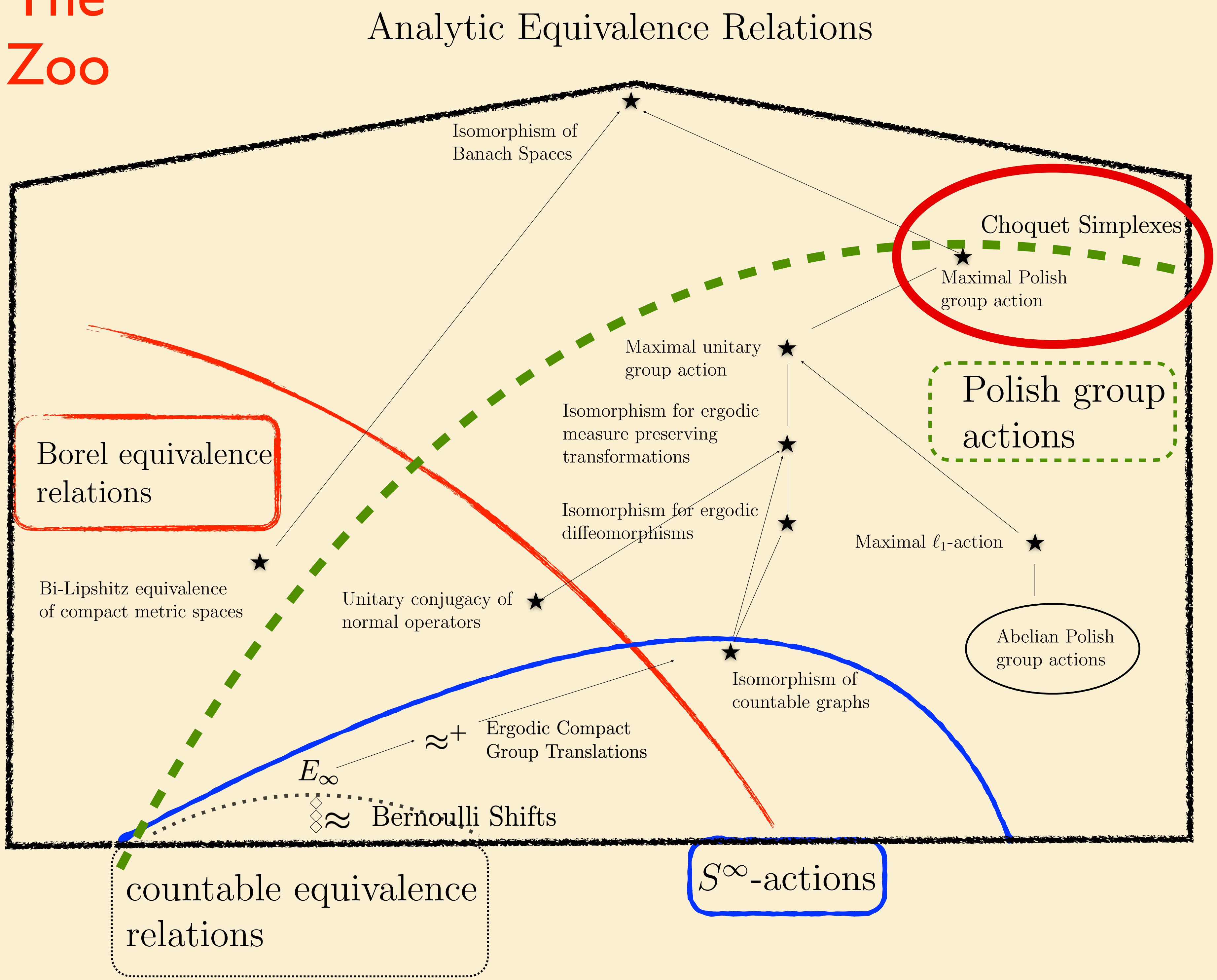
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A RECENT THEOREM FROM 2009

Let $T : [0, 1] \to [0, 1]$ be ergodic

- $[T] = \{S : \text{the orbits of } S \text{ are subsets of the orbits of } T\}$
- $O(T) = \{S : \text{the orbits of } S \text{ are equal to the orbits of } T\}$

Dye proved that every ergodic transformation S is isomorphic to some element $S' \in [T]$.
In 2009, I wrote a note with B. Weiss showing that there is a very constructive map

$$\pi : \text{ergodic } MPT \to O(T)$$

such that $\pi(S) \cong S$. This was in the context of showing that $O(T)$ has the same generic collections of transformations as MPT.

A RECENT THEOREM FROM 2009

It was easy to check that the resulting map is Borel when $[T]$ is given the uniform topology. Hence

$$\text{(isomorphism for ergodic MPTs)} \quad \preceq_{\mathcal{B}} \quad \text{isomorphism for members of } O(T)$$

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The witnesses to isomorphism are not in [T]

SABOK'S THEOREM

Facts

- The Poulsen simplex is universal: every Choquet simplex is affinely homeomorphic to a face of the Poulsen simplex.
- The equivalence relation on Choquet simplexes of being affinely homeomorphic is given by a Polish Group action.

SABOK'S THEOREM

Facts

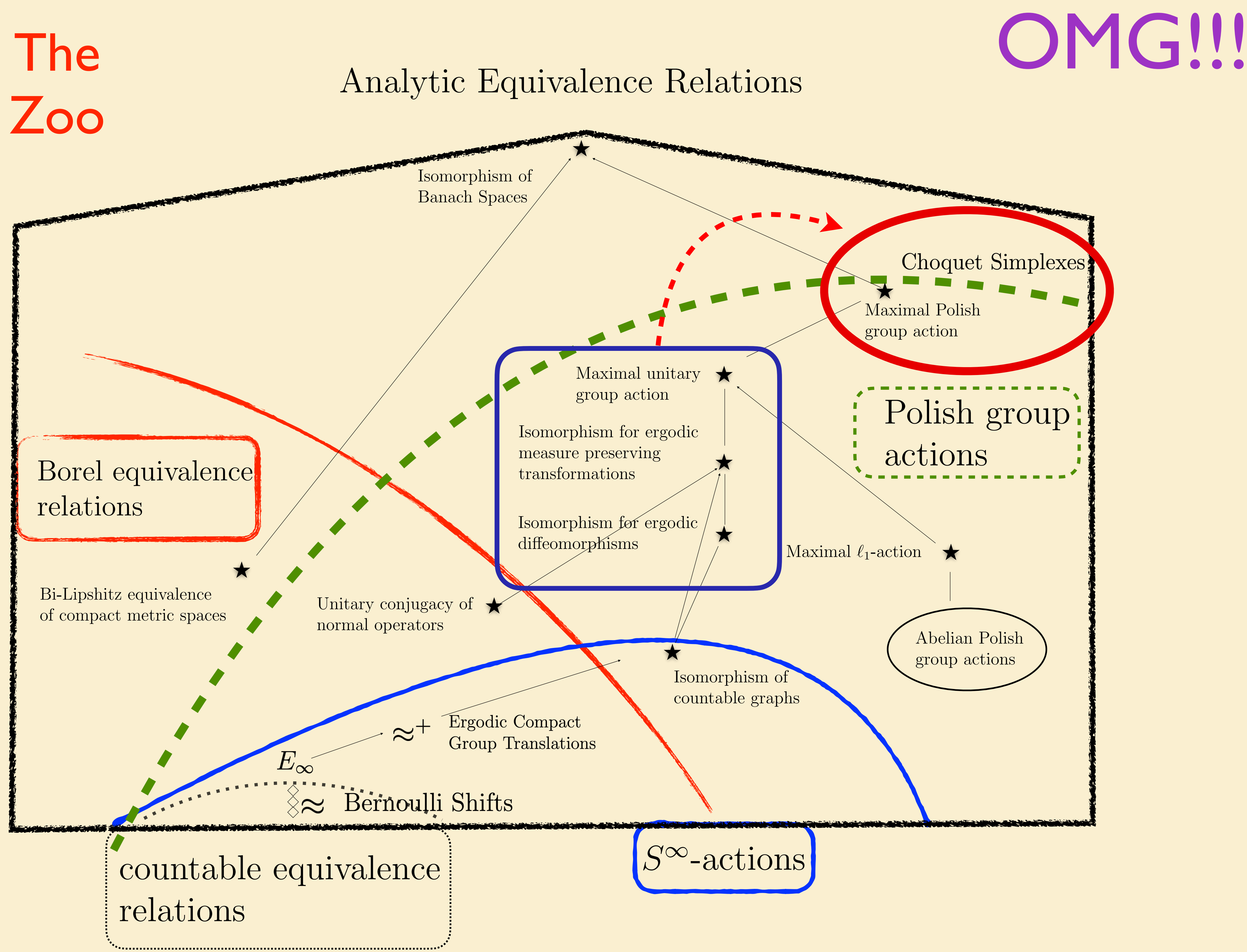
- The Poulsen simplex is universal: every Choquet simplex is affinely homeomorphic to a face of the Poulsen simplex.
- The equivalence relation on Choquet simplexes of being affinely homeomorphic is given by a Polish Group action.

Theorem(Sabok) The equivalence relation of being *affinely homeomomorphic* is maximal among Polish group actions.

SABOK'S CONJECTURE

Theorem(Sabok) The equivalence relation of being *affinely homeomomorphic* is maximal among Polish group actions.

Conjecture(Sabok) The equivalence relation of being *affinely homeomomorphic* Borel reducible to isomorphism for ergodic measure preserving diffeomorphisms of $\mathbb{T}^2$.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THE END
(BUT JUST THE BEGINNING)

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LECTURE 2: GLOBAL STRUCTURE THEORY

Matt Foreman
UC Irvine, August 15, 2023

The author would like to acknowledge support from US NSF Grant DMS-2100367

GOAL OF LECTURE 2 AND 3

Outline a proof of:

Theorem(Foreman, Weiss) Let K be compact metrizable Choquet simplex. Then there is a C^∞ map $T : \mathbb{T}^2 \to \mathbb{T}^2$ such that

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- $M_T(\mathbb{T}^2)$ is affinely homeomorphic to K,
- T is Lebesgue measure preserving and ergodic.

The proof actually gives a quite general Global Structure Theorem for the ergodic measure preserving transformations and their factor structures.

WHAT CAN YOU SAY ABOUT THE *GROUP* MPT AND ITS CONJUGACY ACTION?

IS THERE A STRUCTURAL OBSTACLE TO REPRESENTING EVERY ERGODIC TRANSFORMATION AS A DIFFEOMORPHISM?

MEASURE PRESERVING TRANSFORMATIONS

Consider the group of invertible measure preserving transformations of $[0, 1]$ with the weak topology.

It has various names such as $\mathrm{Aut}(\lambda)$, but we will simply call it MPT.

THE MOST OBVIOUS QUESTION

Is every automorphism of MPT *inner*?

Yes and the group is simple.

Eigen '81, Fathi '78

WHAT OTHER STRUCTURE MIGHT BE RELEVANT?

Objects The Ergodic Transformations $\mathcal{E}$

Structure Factors/Extensions, compactness, mixing properties, invariant measures

WHAT OTHER STRUCTURE MIGHT BE RELEVANT?

Notation for the set of ergodic transformations

Objects The Ergodic Transformations $\mathcal{E}$

Structure Factors/Extensions, compactness, mixing properties, invariant measures

HOMEOMORPHISMS OF $\mathcal{E}$ THAT PRESERVE ISOMORPHISMS AND THE FACTOR PARTIAL ORDERING

Some obvious homeomorphisms are compositions of:

- the map $T \mapsto T^{-1}$
- conjugations: $T \mapsto \phi T \phi^{-1}$.

There are more...

OPEN QUESTION

Is there a non-trivial homeomorphsm $\Phi : \mathcal{E} \to \mathcal{E}$ that preserves isomorphism and the factor partial ordering?

FOCUS OF THIS TALK

Two classes of ergodic transformations

- The *odometer based* transformations
- The *circular systems*

UPSHOT OF THE TALK

Two classes of ergodic transformations

- The the odometer based transformations encode essentially all of the structure of factors, simplexes of invariant measures, distal height, joinings . . . ,
- The circular systems are realizable as Lebesgue measure preserving diffeomorphisms of $\mathbb{T}^2$,
- They form two functorial isomoorphic categories.

THIRD TALK

- Circular systems can be realized as diffeomorphisms in a manner that preserves the simplex of invariant measures.

DIAGRAM OF THE CONSTRUCTION

Downarowicz Toeplitz construction

↓

An odometer based system **O**

↓

The transformation of **O** into a circular system **C**

↓

Realizing **C** in a manner that preserves the simplex

DIAGRAM OF THE CONSTRUCTION

A small complaint: Each step is a paper or two that are 30-75 pages

Downarowicz Toeplitz construction

↓

An odometer based system **O**

↓

The transformation of **O** into a circular system **C**

↓

Realizing **C** in a manner that preserves the simplex

SPECIFICALLY RELEVANT TODAY

- Downarowicz: The Choquet simplex of invariant measures for minimal flows, *Israel Journal of Mathematics*, 1991
- Foreman, Weiss: Representing Anosov-Katok systems, *Journal d'Analyse Mathmatique*, 2015
- Foreman, Weiss: From odometers to circular systems: a global structure theorem, *Journal of Modern Dynamics*, 2017
- Foreman, Weiss: Measure preserving diffeomorphisms of the torus are unclassifiable, *Journal European Math Society*, 2022
- Foreman, Weiss: Odometer Based Systems, *Israel Journal of Mathematics*, 2020

ODOMETER TRANSFORMATIONS

Fix a sequence of integers $\langle k_i : i \in \mathbb{N} \rangle$.

- Let $\mathcal{O} = \prod_{i \in \mathbb{N}} \mathbb{Z}_{k_i}$.
- Then $\mathcal{O}$ is a compact abelian group, so has Haar measure, μ.
- Let $\mathbf{1} = (1, 0, 0, 0 \ldots)$. Then the sums of $\mathbf{1}$ are dense in $\mathcal{O}$
- Define $T(\vec{x}) = \vec{x} + \mathbf{1}$.

Then $(\mathcal{O}, \mathcal{B}, \mu, T)$ is an ergodic measure preserving system.

CONSEQUENCES OF HALMOS-VON NUEMANN THEOREM

- Every odometer transformations has discrete spectrum, and the eigenvalues of the Koopman operator are products of the $e^{2\pi i/k_i}$'s
- If $T \in MPT$ is ergodic and it has infinitely many eigenvalues of finite order, then it contains an odometer factor
- If $T \in MPT$ is ergodic and does not have an odometer factor, then there is an odometer $\mathcal{O}$ such that $T \times \mathcal{O}$ is ergodic.

CONSEQUENCES OF HALMOS-VON NUEMANN THEOREM

- The factor structure of $T \times \mathcal{O}$ can be understood explicitly from the factor structure of T
- For an arbitrary ergodic S the joining structure of $T \times \mathcal{O}$ with S can be understood explicitly from the joining structure of T with S and the eigenvalues of the Koopman operator associated with S.

Definition A *construction sequence* in a finite alphabet Σ is a sequence of non-empty collections of words $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ with the properties that:

1. $\mathcal{W}_0 = \Sigma$,
2. all of the words in each $\mathcal{W}_n$ have the same length q_n and the collection $\mathcal{W}_n$ is uniquely readable,
3. each $w \in \mathcal{W}_n$ occurs at least once as a subword of every $w' \in \mathcal{W}_{n+1}$,
4. there is a summable sequence $\langle \epsilon_n : n \in \mathbb{N} \rangle$ of positive numbers such that for each n, every word $w \in \mathcal{W}_{n+1}$ can be uniquely parsed into segments

$$u_0 w_0 u_1 w_1 \ldots w_l u_{l+1} \tag{1}$$

such that each $w_i \in \mathcal{W}_n$, $u_i \in \Sigma^{<q_n}$ and for this parsing

$$\frac{\sum_i |u_i|}{q_{n+1}} < \epsilon_{n+1}. \tag{2}$$

We call the elements of $\mathcal{W}_n$ "n-words," and let $s_n = |\mathcal{W}_n|$.

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The u's are called "spacers"

$$u_0 w_0 u_1 w_1 \ldots w_l u_{l+1} \tag{1}$$

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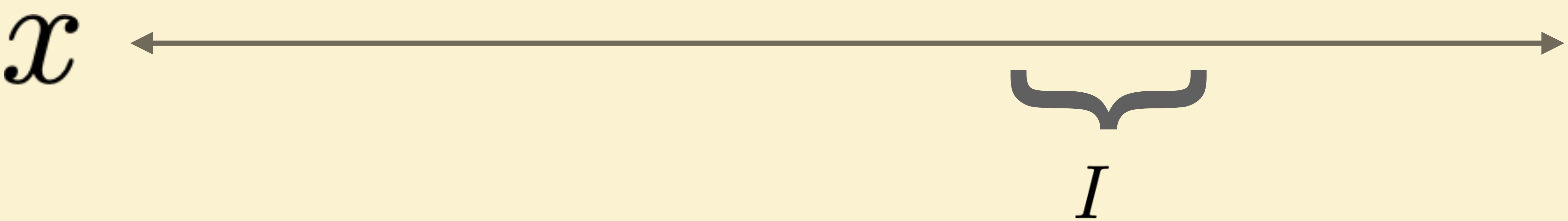
We call the elements of $\mathcal{W}_n$ "n-words," and let $s_n = |\mathcal{W}_n|$.

LIMITS OF CONSTRUCTION SEQUENCES

- Let $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ be a construction sequence in an alphabet Σ. The limit of $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is defined to be the collection $\mathbb{K}$ of $x \in \Sigma^{\mathbb{Z}}$ such that for all finite intervals $I \subseteq \mathbb{Z}$ there is a $w \in \mathcal{W}_n$ and $J \subseteq [0, q_n - 1)$ for some n such that $x \upharpoonright I = w \upharpoonright J$.

- Suppose $x \in \mathbb{K}$ is such that for some $a_n \leq 0 < b_n$ and $x \upharpoonright [a_n, b_n) \in \mathcal{W}_n$. Then $w = x \upharpoonright [a_n, b_n)$ is the *principal n-subword* of x.

The limit of $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is defined to be the collection $\mathbb{K}$ of $x \in \Sigma^{\mathbb{Z}}$ such that for all finite intervals $I \subseteq \mathbb{Z}$ there is a $w \in \mathcal{W}_n$ and $J \subseteq [0, q_n - 1)$ for some n such that $x \upharpoonright I = w \upharpoonright J$.



$$x \upharpoonright I = w \upharpoonright J$$

ODOMETER BASED CONSTRUCTION SEQUENCES

A construction sequence is *odometer based* if there are no spacers:

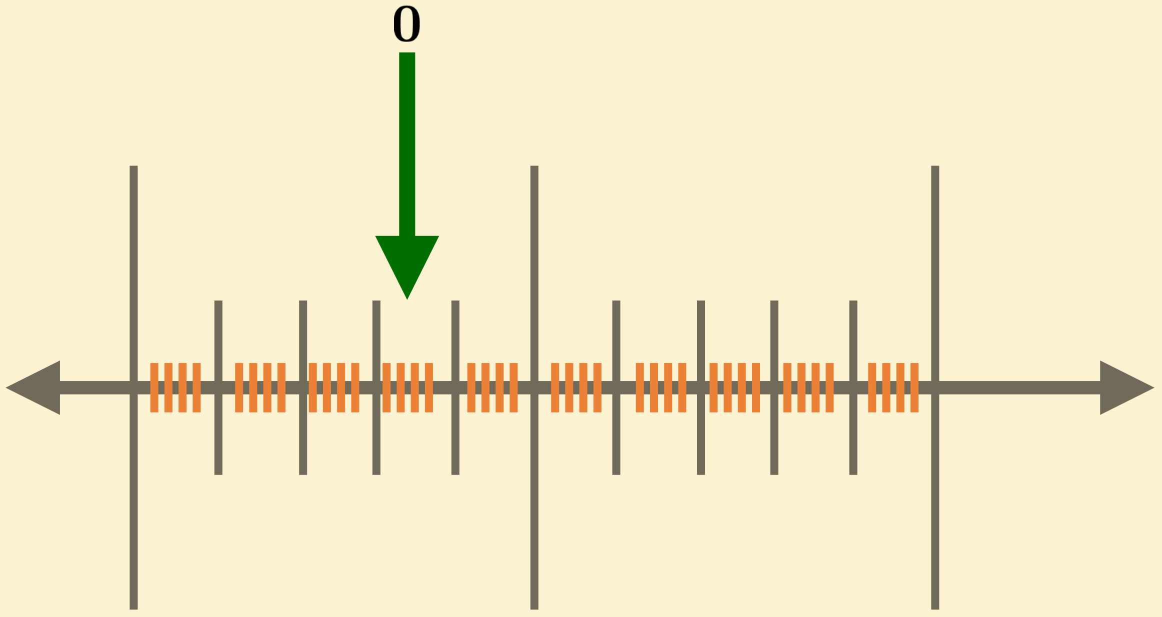
$$\mathcal{W}_{n+1} \subseteq (\mathcal{W}_n)^{k_n}$$

for some k_n.

Definition An odometer based system is a subshift that is a limit of an odometer based construction sequence.

ODOMETER BASED SYSTEMS HAVE A CANONICAL ODOMETER FACTOR

Let $\mathbb{K}$ be an odometer based system. The k^{th} value in the odometer is given by where 0 is in the principal subword.



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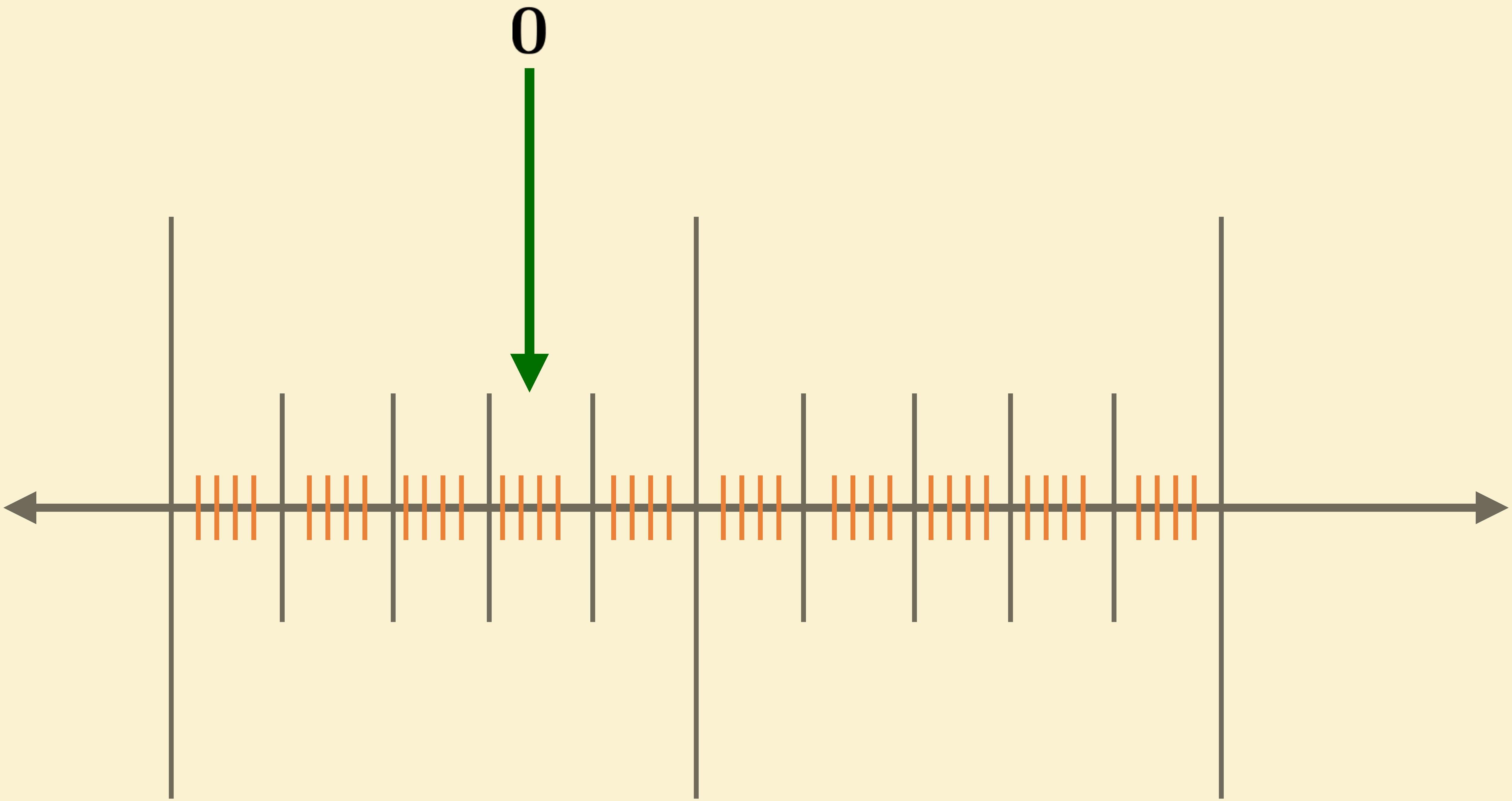

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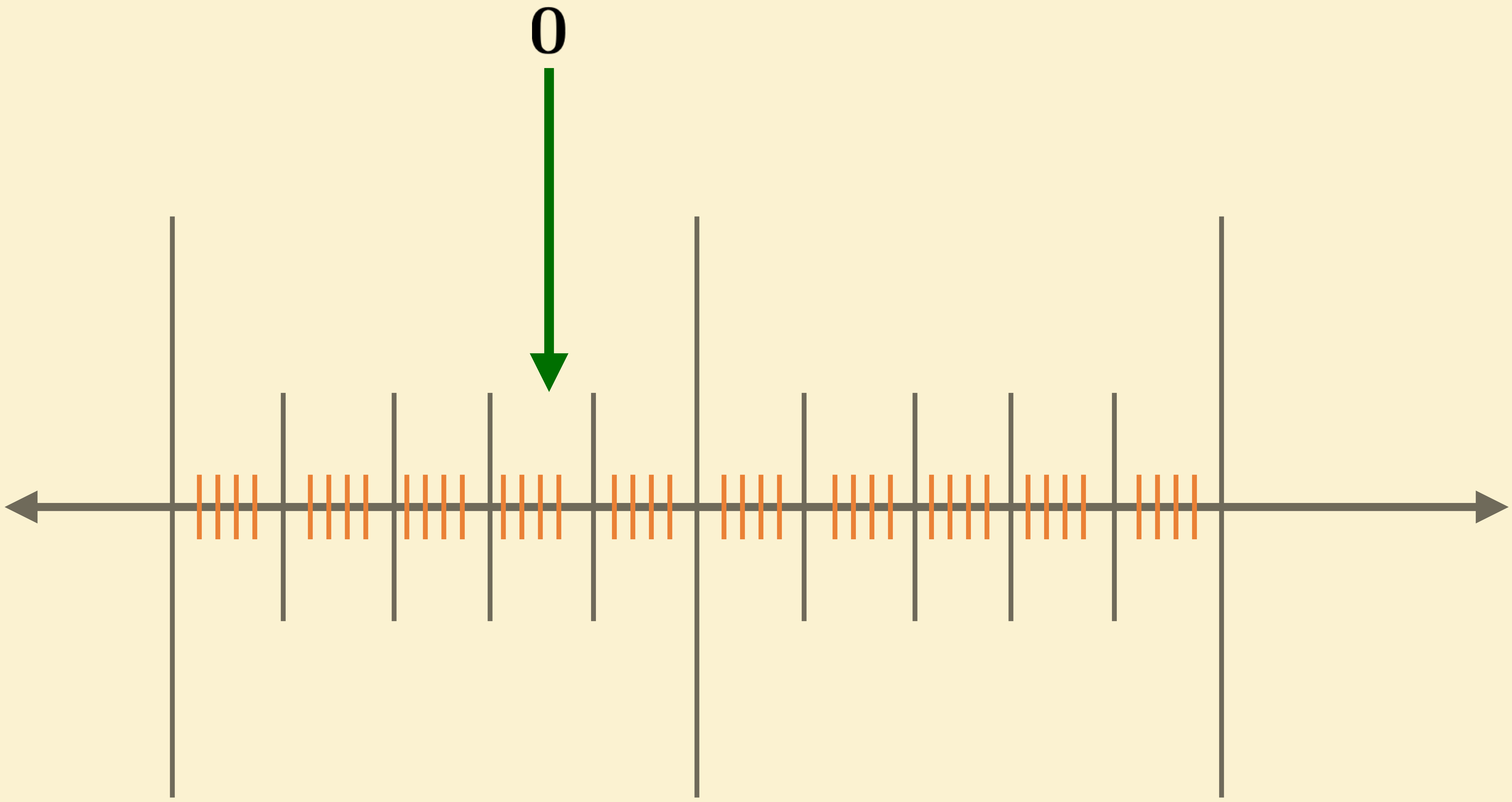

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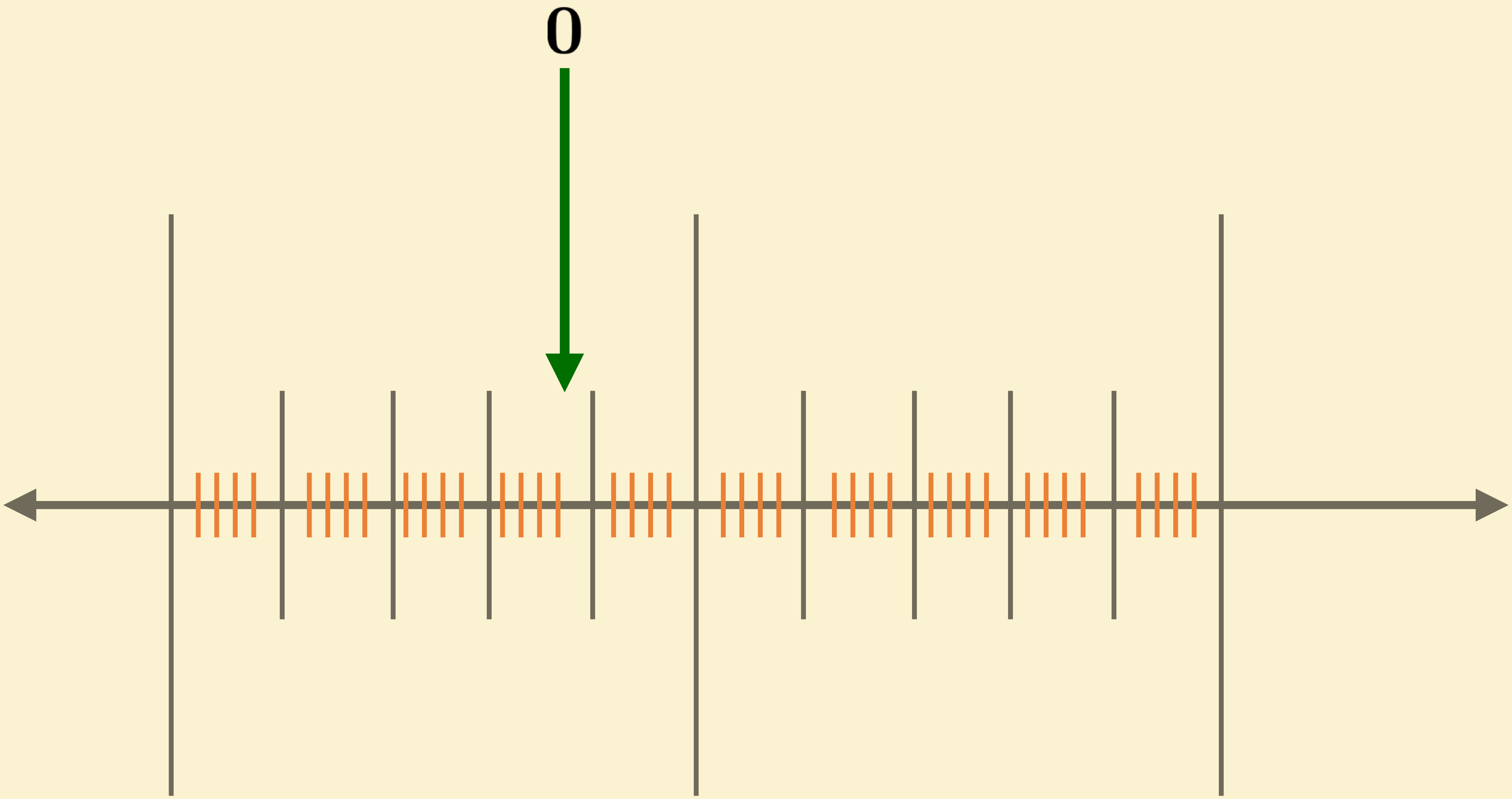

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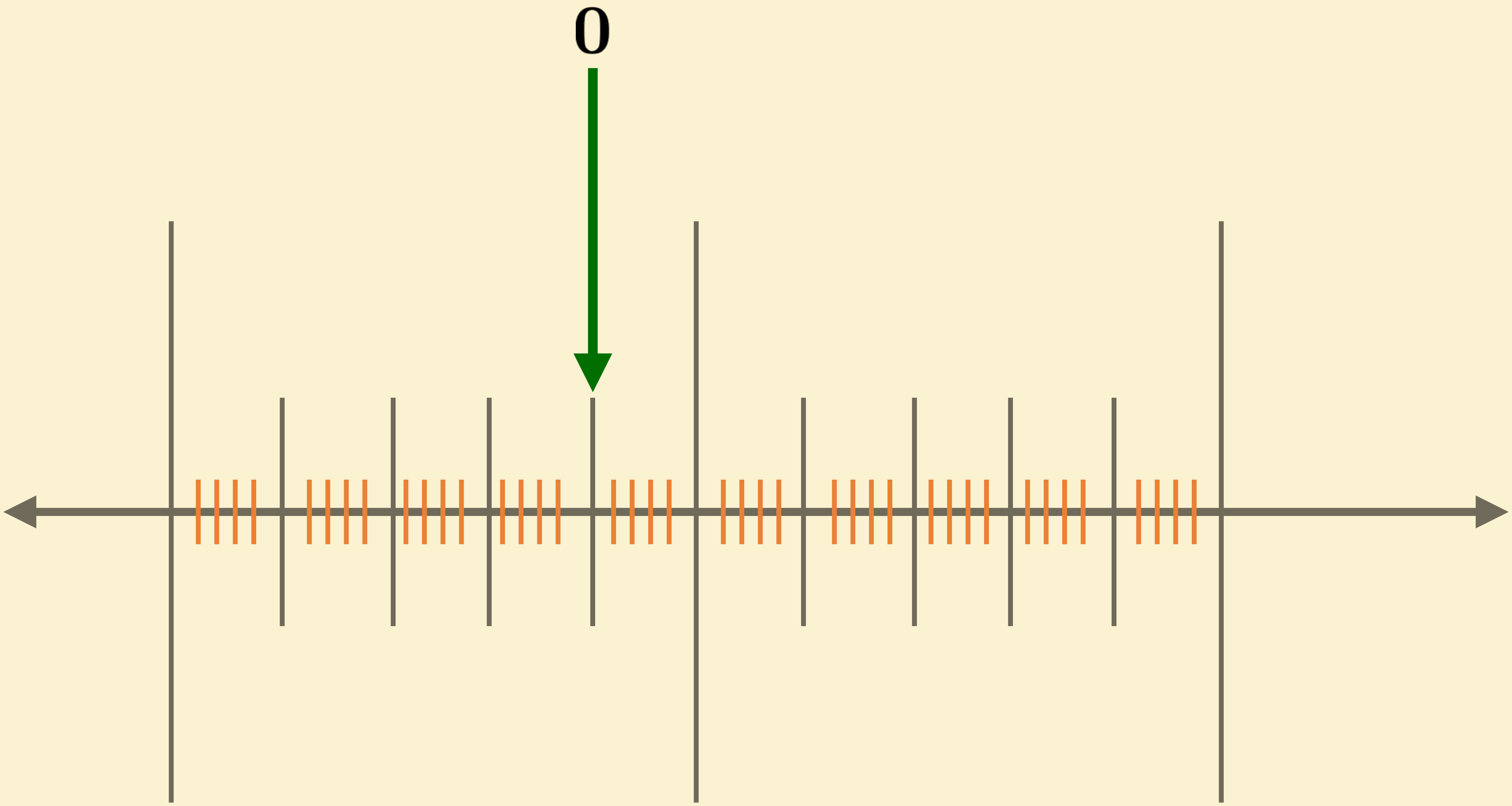

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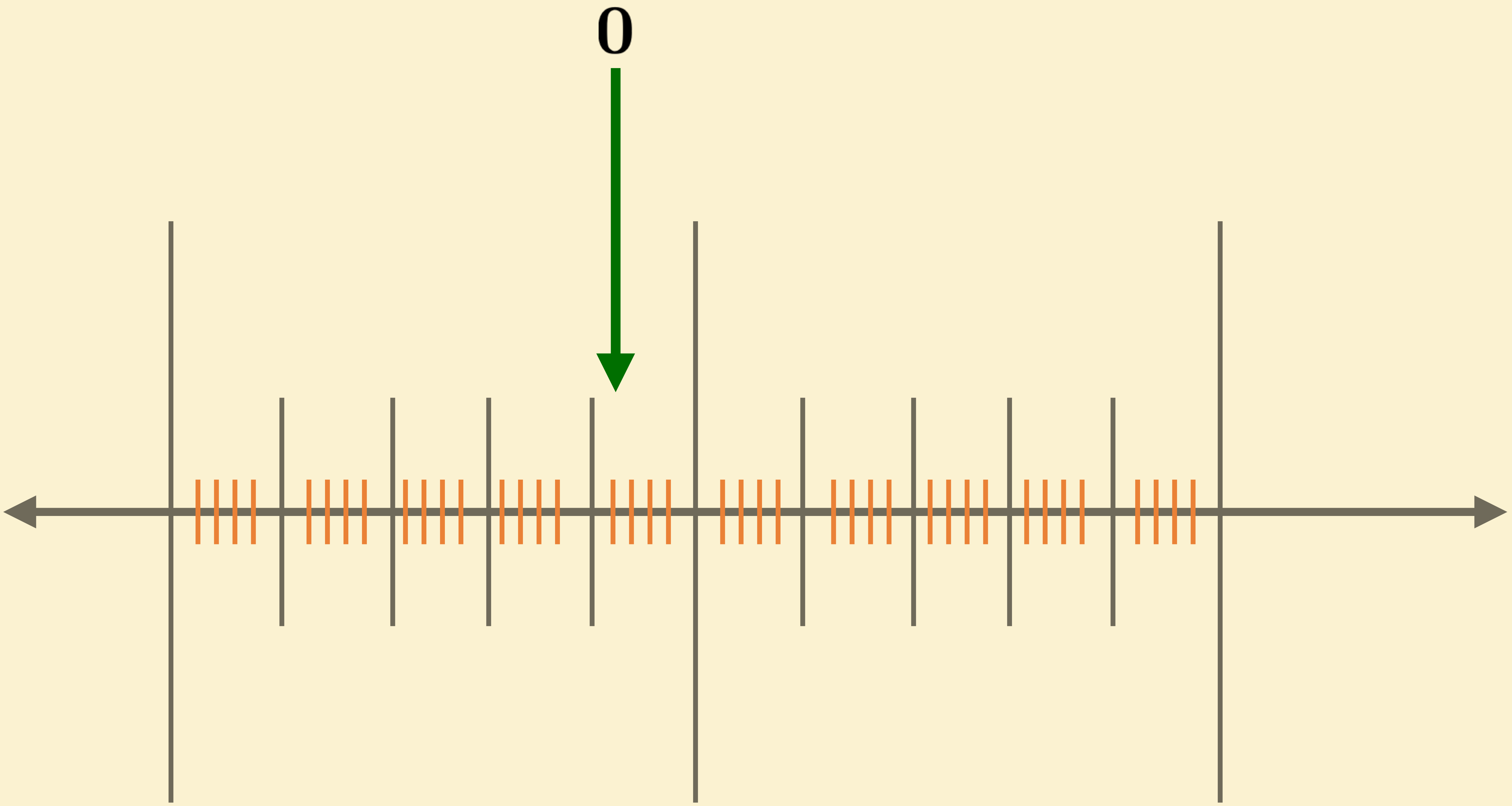

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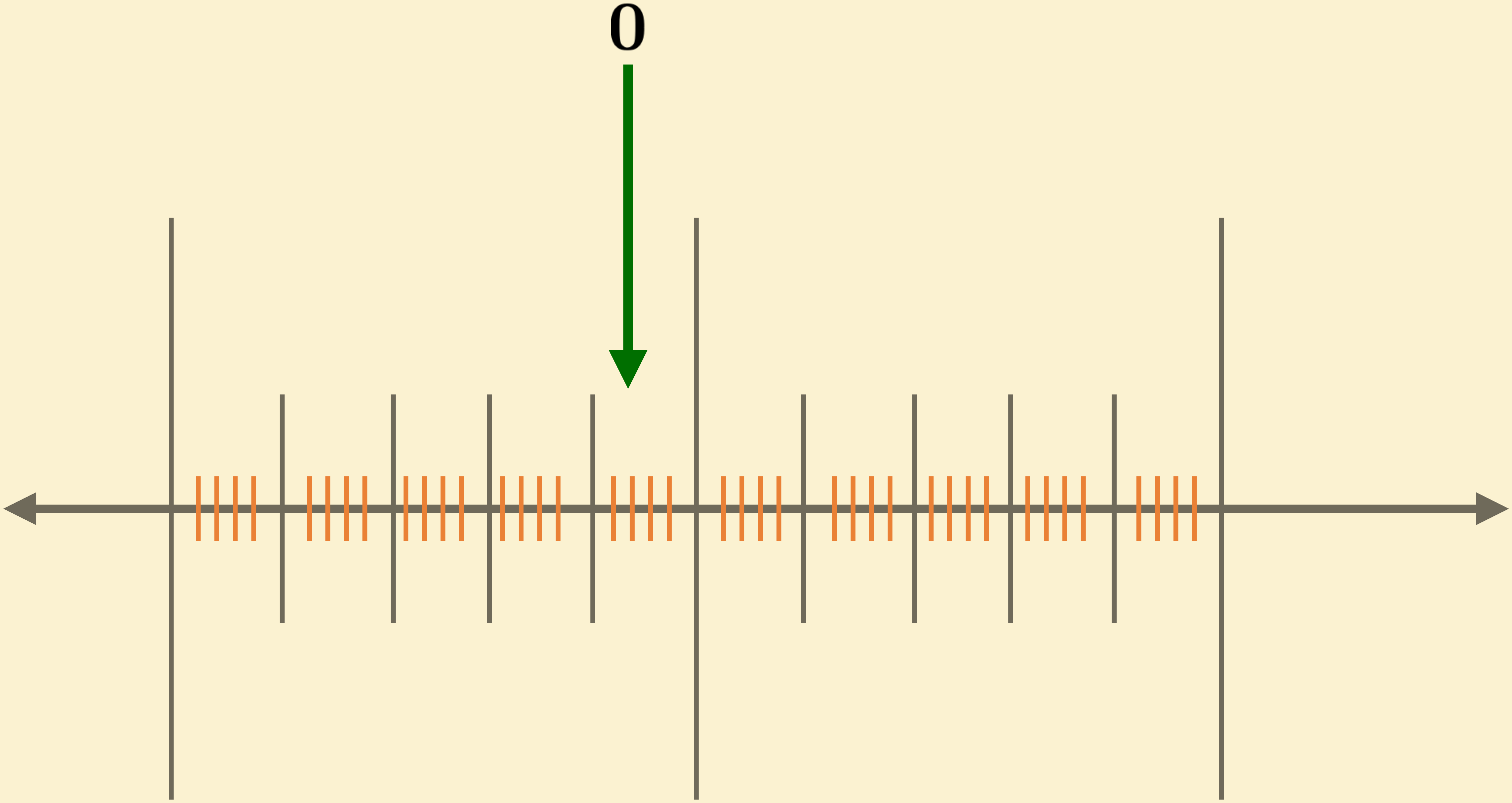

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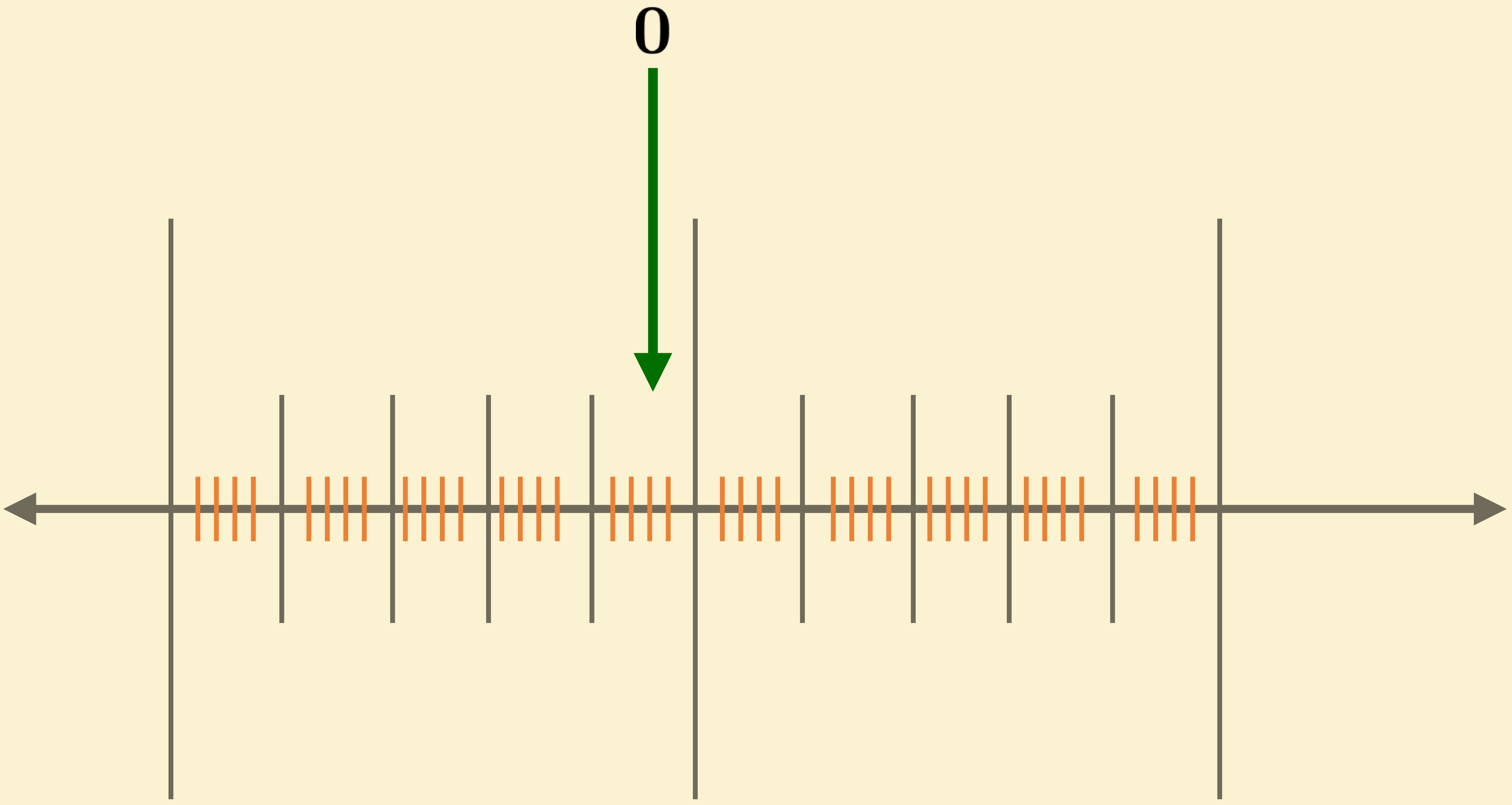

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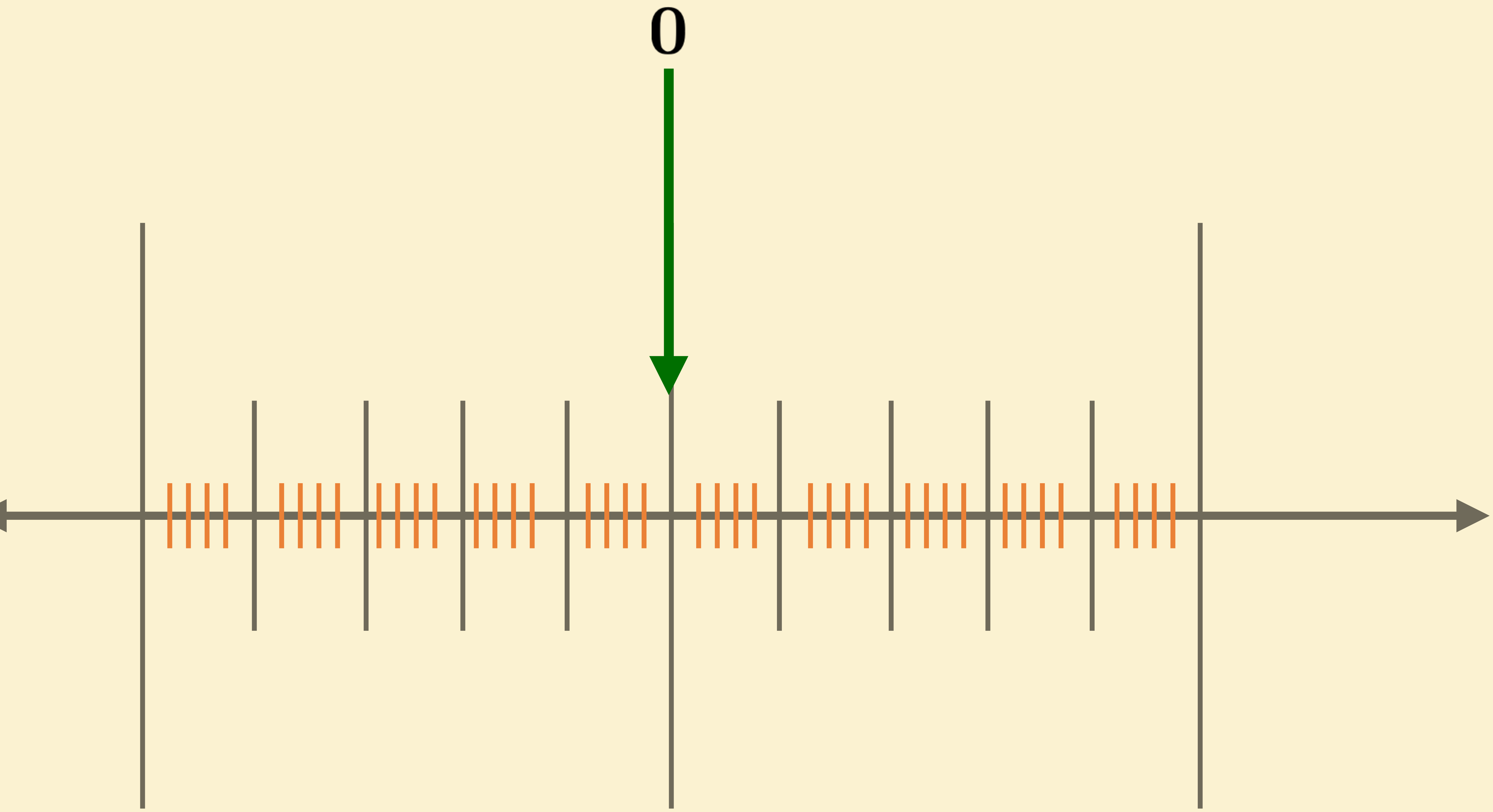

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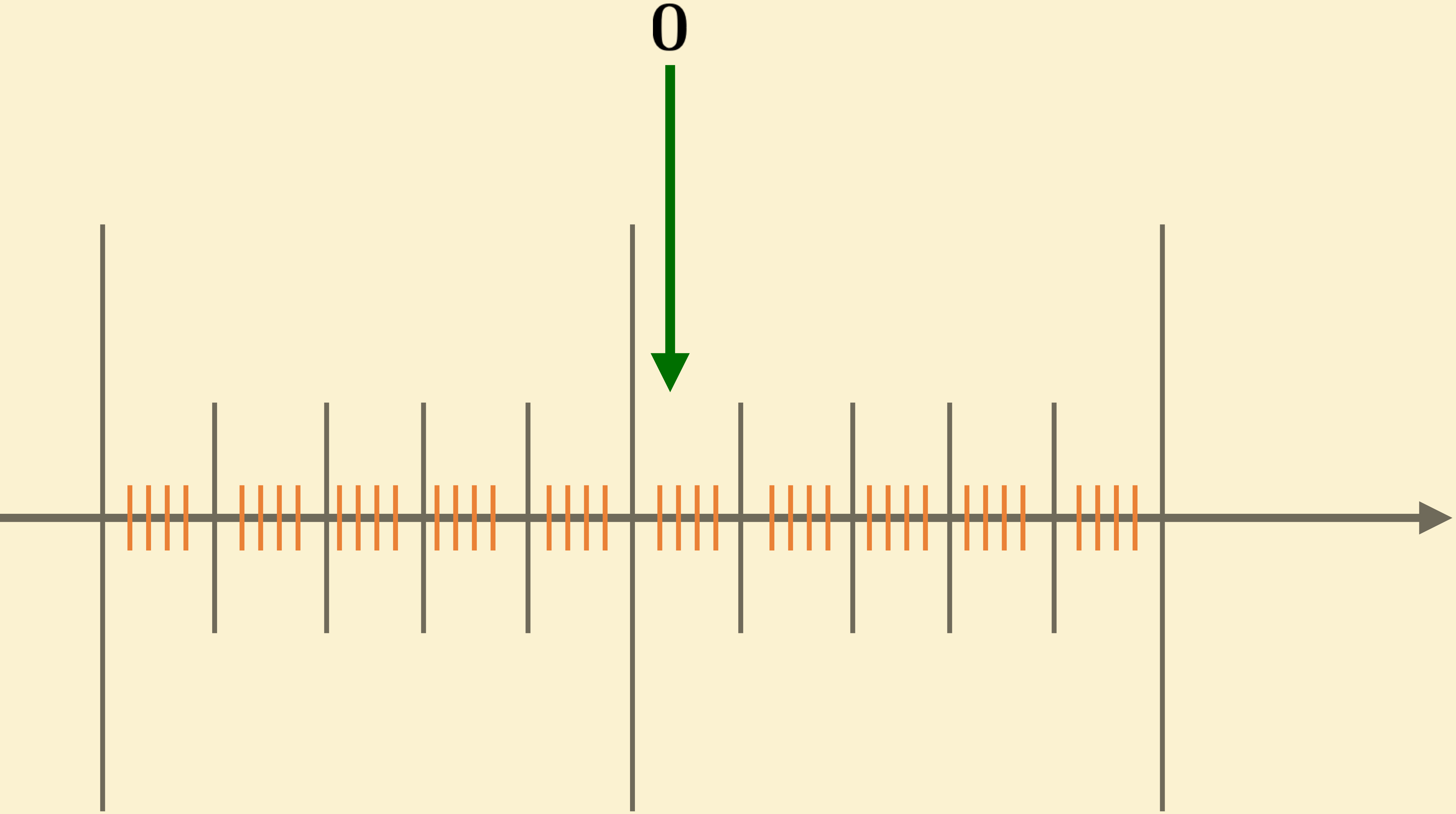

PROPERTIES OF K

- Suppose that $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is an odometer based construction sequence for a symbolic system $\mathbb{K}$. Let K_n be the length of the words in $\mathcal{W}_n$, $k_0 = K_1$ and for $n > 0$, $k_n = K_{n+1}/K_n$. Then the odometer $\mathfrak{O}$ determined by $\langle k_n : n \in \mathbb{N} \rangle$ is canonically a factor of $\mathbb{K}$.

- $\mathbb{K}$ can be constructed to be a topologically minimal subshift.

THE POINT

Theorem Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system with finite entropy. Then X has an odometer factor if and only if X is measure isomorphic to a topologically minimal odometer based symbolic system.

LOOKING' GOOD EH??

- Downarowicz' construction builds a Toeplitz sequence who's simplex of invariant measures is any given K
- Toeplitz sequences have odometer factors
- Downarowicz' Toeplitz sequences are isomorphic to Odometer based transformations
- ?? One can copy over Downarowcz simplex to the simplex on the associated odometer based system.??

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- Toeplitz sequences have odometer factors
- Downarowicz' Toeplitz sequences are isomorphic to Odometer based transformations
- ?? One can copy over Downarowcz simplex to the simplex on the associated odometer based system.??

We were lucky ….

YEAH YEAH …

Definition Let (Z, σ, S) and (X, τ, T) be minimal compact topological systems and $\pi : Z \to X$ be a continuous factor map. Then (π, Z) is an *augmentation* of X if there is an S-invariant Borel set $A \subseteq Z$ such that if

$$L = \{x : \text{there is exactly one } y \in A \text{ with } \pi(y) = x\},$$

then for all T-invariant μ on X, $\mu(L) = 1$.

AUGMENTATIONS

Proposition Suppose that (π, Z) is an augmentation of X. Then there is a canonical affine homeomorphism of $\mathcal{M}(Z, S)$ with $\mathcal{M}(X, T)$.

UPSHOT

Proposition Let $\mathbb{L}$ be the orbit closure of a Toeplitz sequence x, $\mathfrak{O}$ be its maximal odometer factor based on a sufficiently fast growing sequence $\langle k_n \rangle$ and $\mathbb{K}$ be a canonical odometer based presentation of $\mathfrak{O}$. Then there is an odometer based system $\mathbb{L}^* \subseteq \mathbb{L} \times \mathbb{K}$ such that if $\pi : \mathbb{L}^* \to \mathbb{L}$ is the projection to the first coordinate, then $(\pi, \mathbb{L}^*)$ is an augmentation of $\mathbb{L}$.

IN ENGLISH

Proposition Given a metrizable Choquet simplex K, then there is an odometer based system that has K as its simplex of invariant measures.

SO WHAT??

It isn't known how to find a diffeomorphism of a compact manifold that has an odometer factor!

How is this even helpful?

A PRE-EXISTING THEOREM

There are two categories of measure preserving systems:

- $\mathcal{O}B$, the collection of ergodic odometer based systems
- $\mathcal{C}$, the collection of circular systems.

A PRE-EXISTING THEOREM

There are two categories of measure preserving systems:

- $\mathcal{O}B$ contains "most" measure preserving systems. It's structure reflects all behavior of joinings, extensions, invariant simplexes of measures, relatively distal extensions ...

  Odometer Based systems

- $\mathcal{C}$ is a class of symbolic systems that can be realized as diffeomorphisms of $\mathbb{T}^2$.

  Circular Systems

WHY ARE "MOST" TRANSFORMATIONS ODOMETER BASED?

Consider ergodic measure preserving transformations T ordered by setting $S \preceq T$ if S is a factor of T. Then $\{T : T \text{ is odometer based }\}$ is a *cone*:

a.) If T is odometer based and $T \preceq T'$ then T' is odometer based,

b.) If T is not odometer based then there is an odometer $\mathcal{O}$ such that $T' = T \times \mathcal{O}$ is ergodic.

INFORMAL STATEMENT

There are two categories of measure preserving systems:

- $\mathcal{O}B$ contains "most" measure preserving systems. It's structure reflects all behavior of joinings, extensions, invariant simplexes of measures, relatively distal extensions . . .
- $\mathcal{C}$ is a class of symbolic systems that can be realized as diffeomorphisms of $\mathbb{T}^2$.

The Global Structure Theorem says the two categories are functorial isomorphic. It follows that for every metrizable Choquet simplex there is a circular system with that simplex of measures.

The Catch: The smooth realization of the circular systems have to preserve the simplex of invariant measures.

THE PLAN

The rest of this lecture will be devoted to a rigorous statement of the Global Structure Theorem. Lecture 3 will be description of how to modify that Anosov-Katok method to realize the circular system preserving the collection of all invariant measures.

CIRCULAR SYSTEMS

Circular systems are built using the "Circular Operator" that has parameters $\langle (k_n, l_n) : n \in \mathbb{N}\rangle$. These are used (in the fashion of Anosov and Katok) to build a sequences of natural numbers $\langle (p_n, q_n) : n \in \mathbb{N}\rangle$:

- $p_0 = 0, q_0 = 1,$
- Inductively set:

$$\begin{aligned} q_{n+1} &= k_n l_n q_n^2 \\ p_{n+1} &= p_n q_n k_n l_n + 1. \end{aligned}$$

- $\alpha_n = \frac{p_n}{q_n}$.

WHAT'S THE POINT?

- $p_0 = 0, q_0 = 1,$
- Inductively set:

$$\begin{aligned} q_{n+1} &= k_n l_n q_n^2 \\ p_{n+1} &= p_n q_n k_n l_n + 1. \end{aligned}$$

- $\alpha_n = \frac{p_n}{q_n}$.

Then $(p_n, q_n) = 1$ and $\alpha_{n+1} = \alpha_n + \frac{1}{q_{n+1}}$

The $2\pi\alpha_n$ codes a rotation of the unit circle and by taking l_n very large the $n+1^{st}$ rotation is arbitrarily close to the n^{th} rotation.

ONE MORE NUMBER

Since $(p_n, q_n) = 1$, we can define

$$j_i = (p_n)^{-1} i \pmod{q_n}$$

Fix a collection of symbols Σ and let $\{b, e\}$ be two more. Let $w_0, \ldots w_{k_n-1}$ be words. Define the circular operator $\mathcal{C}$ by setting:

$$\mathcal{C}(w_0, w_1, w_2, \ldots w_{k-1}) = \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{q-j_i} w_j^{l-1} e^{j_i})$$

Raising a letter or a word to a power means repeated concatenation .....

Let Σ be a non-empty finite or countable alphabet. We will construct the systems we study by building collections of words $\mathcal{W}_n$ in the alphabet $\Sigma \cup \{b, e\}$ by induction as follows:

- Fix a circular coefficient sequence $\langle k_n, l_n : n \in \mathbb{N}\rangle\rangle$.
- Set $\mathcal{W}_0 = \Sigma$.
- Having built $\mathcal{W}_n$ we choose a set $P_{n+1} \subseteq (\mathcal{W}_n)^{k_n}$ and form $\mathcal{W}_{n+1}$ by taking all words of the form $\mathcal{C}(w_0, w_1 \dots w_{k_n-1})$ with $(w_0, \dots w_{k_n-1}) \in P_{n+1}$.

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The result is a *circular construction sequence.*

WHY "CIRCULAR" ???

If $\Sigma = \{*\}$ is a set with just one symbol, then the limit $\mathbb{K}^c$ of the circular construction sequence is conjugate to a rotation of the unit circle by

$$\alpha = \lim_n \alpha_n.$$

WHY "CIRCULAR" ???

If $\Sigma = \{*\}$ is a set with just one symbol, then the limit $\mathbb{K}^c$ of the circular construction sequence is conjugate to a rotation of the unit circle by

$$\alpha = \lim_n \alpha_n.$$

It follows that every circular system has a factor that is a rotation of the circle. By taking the sequence of l_n's to grow fast enough the rotation is by a Liouvillean irrational number.

A TRIVIAL DEFINITION

For a fixed subshifts $\mathcal{S} = \Sigma^{\mathbb{Z}}, \mathcal{T} = \Gamma^{\mathbb{Z}}$, a map $f : \Sigma^{\mathbb{Z}} \to \Gamma^{\mathbb{Z}}$ is

- *synchronous* if is a factor map from $\mathcal{S}$ to $\mathcal{T}$,
- *anti-synchronous* if it is a factor map from $\mathcal{S}$ to $(\mathcal{T})^{-1}$.

TWO CATEGORIES

Fix an arbitrary circular coefficient sequence $\langle k_n, l_n : n \in \mathbb{N} \rangle$ for the rest of the Lecture.

TWO CATEGORIES

Let $\mathcal{O}B$ be the category

- whose objects are ergodic odometer based systems with coefficients $\langle k_n : n \in \mathbb{N} \rangle$.
- Whose morphisms between objects $(\mathbb{K}, \mu)$ and $(\mathbb{L}, \nu)$ will be synchronous graph joinings of $(\mathbb{K}, \mu)$ and $(\mathbb{L}, \nu)$ or anti-synchronous graph joinings of $(\mathbb{K}, \mu)$ and $(\mathbb{L}^{-1}, \nu)$.

We call this the *category of odometer based systems.*

TWO CATEGORIES

Let $\mathcal{C}B$ be the category

- whose objects consists of ergodic circular systems with coefficients $\langle k_n, l_n : n \in \mathbb{N} \rangle$.
- whose morphisms between objects $(\mathbb{K}^c, \mu^c)$ and $(\mathbb{L}^c, \nu^c)$ will be synchronous graph joinings of $(\mathbb{K}^c, \mu^c)$ and $(\mathbb{L}^c, \nu^c)$ or anti-synchronous graph joinings of $(\mathbb{K}^c, \mu^c)$ and $((\mathbb{L}^c)^{-1}, \nu^c)$.

We call this the *category of Circular systems.*

THE THEOREM

Theorem (F-W) For a fixed circular coefficient sequence $\langle k_n, l_n : n \in \mathbb{N}\rangle$ the categories $\mathcal{O}B$ and $\mathcal{C}B$ are isomorphic by a functor $\mathcal{F}$ that takes synchronous joinings to synchronous joinings, anti-synchronous joinings to anti-synchronous joinings, isomorphisms to isomorphisms and compact and weakly mixing extensions to compact and weakly mixing extensions.

THE THEOREM

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It follows that F is a preserves isomorphism and non-isomorphism—it is a reduction!!

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- $\mathcal{F}$ is a functor: it preserves compositions.
- $\mathcal{F}$ preserves the word statistics: frequencies of $\mathcal{W}_n$-words in $\mathcal{W}_{n+1}$-words, relative measures etc.
- So: $\mathcal{F}$ preserves the simplex of invariant measures.
- If follows that $\mathcal{F}$ preserves facts like measure-distality (generalized discrete spectrum)
- Moreover, it preserves distal rank (more later on this).
- etc. etc. etc. THE TWO CATEGORIES ARE THE SAME!

NEXT TIME:
THINGS GET TECHNICAL

THE END

https://img.theculturetrip.com/1440x807/smart/wp-content/uploads/2017/03/shutterstock_481584733-tomasz-mazon.jpg

THE SIMPLEX OF MEASURES INVARIANT UNDER DIFFEOMORPHISMS
LECTURE 3

Matt Foreman
UC Irvine, August 17, 2023

The author would like to acknowledge support from US NSF Grant DMS-2100367

APPLICATIONS OF THE GLOBAL STRUCTURE THEOREM

Theorem (Foreman-Weiss) Let X be the space of Lebesgue measure preserving C^∞-diffeomorphisms of $\mathbb{T}^2$. Let E be the equivalence relation of being conjugate by a measure preserving transformation. Then E is complete analytic, in particular it is not Borel.

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- *Graph Isomorphism* can be reduced to E.

  So E is S-infty complete

- Essentially the same proof works for the equivalence relation of *flip conjugacy*.

FURSTENBERG'S CLASSIFICATION

Theorem Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system. Then there is a countable ordinal η and a system of measure-preserving transformations $\langle (X_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha) : \alpha \leq \eta + 1 \rangle$ such that

1. $(X_0, \mathcal{B}_0, \mu_0, T_0)$ is the trivial flow.
2. For each $\alpha < \eta$, $X_{\alpha+1}$ is a compact extension of X_α.
3. If α is a limit ordinal then X_α is the inverse limit of $\langle X_\beta : \beta < \alpha \rangle$
4. $X_{\eta+1}$ is either:
   - a trivial extension of X_η (so $X_{\eta+1} \cong X_\eta$), or
   - a weakly-mixing extension of X_η.

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Definition An ergodic measure-preserving transformation is *measure-distal* if there is not weakly-mixing extension at η.

DISTAL RANK

- **Theorem**(Beleznay-Foreman) There are measure distal transformation of all countable ranks.
- **Theorem**(Mary Rees) Any topologically distal diffeomorphism has rank ≤ 3.

USING THE GLOBAL STRUCTURE THEOREM

Theorem (Foreman-Weiss) For every ordinal $\alpha < \omega_1$ there are minimal measure distal C^∞-diffeomorphisms of $\mathbb{T}^2$ of height α.

SO:

GENERAL ERGODIC MPTS (AND DIFFEOMORPHISMS) ARE NOT CLASSIFIABLE …

GENERAL ERGODIC MPT'S ARE NOT CLASSIFIABLE …

WHAT ABOUT SPECIFIC CLASSES: E.G. WEAKLY MIXING TRANSFORMATIONS.

Both abstract MPTs and diffeomorphisms…

HIGHLIGHTS

The following were proved generalizing our techniques

1. **Theorem** (Gerber-Kunde) The Kakutani equivalence relation between diffeomorphisms of the $\mathbb{T}^2$ is complete analytic.
2. **Theorem** (Gerber-Kunde) The conjugacy relation for diffeomorphisms of tori of dimension at least 5 that are $\mathcal{K}$-automorphisms is complete analytic. So is the Kakutani equivalence relation on the $\mathcal{K}$-automorphisms.
3. **Theorem** Same results for weakly mixing of zero entropy in dimensions at least 2.
4. OPEN: (Strongly) mixing transformations of zero entropy.

HIGHLIGHTS

1. **Theorem** (Gerber-Kunde) The Kakutani equivalence relation between diffeomorphisms of the $\mathbb{T}^2$ is complete analytic.
2. **Theorem** (Gerber-Kunde) The conjugacy relation for diffeomorphisms of tori of dimension at least 5 that are $\mathcal{K}$-automorphisms is complete analytic. So is the Kakutani equivalence relation on the $\mathcal{K}$-automorphisms.
3. **Theorem** Same results for weakly mixing of zero entropy in dimensions at least 2.
4. OPEN: (Strongly) mixing transformations of zero entropy.

Each of these results require
long and difficult arguments.

TODAY'S MAIN TASK

How do you realize a circular system as a diffeomorphism preserving the simplex of invariant measures?

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How do you realize a circular system as a diffeomorphism preserving the simplex of invariant measures?

Many of the ideas here derive from earlier work of Fayad and Katok who showed how to get a diffeomorphism of the annulus with exactly two invariant ergodic measures.

START WITH ABSTRACTIONS

EMPIRICAL DISTRIBUTIONS

A slightly oversimplified presentation

Let u, v be words in a collection of letters Γ. Suppose that no two instances of u in v overlap. Suppose that $lh(u) \ll lh(v) = n$. Define

$$OCC(u, v) = |\{i < n : v \restriction [i, i + m) = u\}|.$$

The *density of occurrences of* u *in* v is defined to be

$$d(u, v) =_{def} \frac{Occ(u, v)}{n}.$$

Formally the denominator could be taken to be $n - m$, but for $n \gg m$ this makes little difference.

Let $\mathcal{W}$ be a collections of words of the same length m and $w \in \Gamma^{<\mathbb{N}}$ be written as

$$w = u_0 w_0 u_1 w_1 \dots w_J u_{J+1}$$

with $w_i \in \mathcal{W}$ and $\sum lh(u_i) \ll lh(w)$. Then the empirical distribution on $\mathcal{W}$ determined by w is:

$$\mathrm{EmpDist}_k(w)(w') = \frac{|\{0 \leq j \leq J : w_j = w'\}|}{J+1}.$$

(assume that the lengths of the spacers U_i is negligable)

GENERAL (VAGUE) FACT

Suppose that $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is a (uniquely readable) construction sequence in a finite language Σ and $\langle u_n : n \in \mathbb{N} \rangle$ is a sequence of words of increasing length. Then there is a subsequence $\langle u_{n_i} : i \in \mathbb{N} \rangle$ such that the empirical distributions of the u_{n_i}'s converge to a shift invariant-measure on the limit $\mathbb{K}$ of $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$. The sequence $\langle u_{n_i} : i \in \mathbb{N} \rangle$ will be called *generic* for μ.

CONSEQUENCES OF THE ERGODIC THEOREM

Definition Let $\mathbb{K} \subseteq \Sigma^{\mathbb{Z}}$ be a closed shift invariant set and μ be a shift invariant ergodic measure on $\mathbb{K}$. If $\vec{x} \in \Sigma^{\mathbb{Z}}$ then $\vec{x}$ is *generic* for μ if and only if:

Whenever $\langle a_n : n \in \mathbb{N} \rangle$ and $\langle b_n : n \in \mathbb{N} \rangle$ are increasing sequences of positive numbers with $a_n + b_n \to \infty$ and $J \in \Sigma^k$ is a finite interval:

$$\mu(\langle J \rangle) = \lim_n d(J, x \restriction [-a_n, b_n)) = \mu(J).$$

By the ergodic theorem μ-a.e. $\vec{x}$ is generic for μ.

WHAT'S THE POINT?

Theorem Fix a construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ with limit $\mathbb{K}$. Let $\mathbb{X} = (X, \mathcal{B}, T, \mu)$ be an ergodic measure preserving system with X a Polish space, and $\Gamma = \{A_\sigma : \sigma \in \Sigma\}$ be a generating partition for $\mathbb{X}$ consisting of Borel sets. Suppose that

1. $\phi : \mathbb{K} \to X$ is a Borel measurable, equivariant map that is one-to-one,

2. $B = \{s \in S : \text{the } (T, \Gamma)\text{-name of } \phi(s) \text{ is not } s\} \subseteq \mathbb{K}$ has measure zero for every shift invariant measure on $\mathbb{K}$,

Then there is a affine continuous injection from $M_{sh}(\mathbb{K})$ to $M_T(X)$.

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Then there is a affine continuous injection from $M_{sh}(\mathbb{K})$ to $M_T(X)$.

T will be the diffeomorphism we build on the torus.

IN ENGLISH

Given a system $\mathbb{K}$ with a construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ in a language Σ and an ergodic map $T : X \to X$ with a partition $\Gamma = \{A_\sigma : \sigma \in \Sigma\}$ that gives the same words as in the $\mathcal{W}_n$'s. One can copy over every shift-invariant measure on $\mathbb{K}$ to a shift invariant measure on (T, X).

THE HARD PART

Making sure that **every** T-invariant measure on X comes from a measure on $\mathbb{K} = \lim_n \langle \mathcal{W}_n : n \in \mathbb{N} \rangle$.

TEST SEQUENCES

Definition

- Let $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ be a construction sequence with every word in $\mathcal{W}_n$ having length q_n.
- For each n, let $\mathcal{I}_n$ be a collection of disjoint sub-intervals of $[0, q_n)$.

Then $\vec{\mathcal{I}} = \langle \mathcal{I}_n : n \in \mathbb{N} \rangle$ is a *test sequence* for $\langle q_n : n \in \mathbb{N} \rangle$ if for some $\rho > 0$:

i.) for all n, $|\bigcup_{\{J \in \mathcal{I}_n\}} J| > \rho q_n$, and

ii.) $\lim_{k \to \infty} |\mathcal{I}_n| / q_n = 0$.

Theorem Fix a construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ with limit $\mathbb{K}$. Fix a test sequence $\vec{\mathcal{I}}$ for $\langle q_n : n \in \mathbb{N}\rangle$. Let $\mathbb{X} = (X, \mathcal{B}, T, \mu)$ be an ergodic measure preserving system with X a Polish space, and $\Gamma = \{A_\sigma : \sigma \in \Sigma\}$ be a generating partition for $\mathbb{X}$ consisting of Borel sets. Suppose that

1. $\phi : \mathbb{K} \to X$ is a Borel measurable, equivariant map that is one-to-one on S,
2. $B = \{s \in S : \text{the } \Gamma\text{-name of } \phi(s) \text{ is not } s\} \subseteq \mathbb{K}$ has measure zero for every shift invariant measure on $\mathbb{K}$,
3. $z \in X$ is (μ, Γ)-generic with (T, Γ)-name z^* and there are increasing positive sequences $\{n_k, a_k, b_k : k \in \mathbb{N}\}$ and words $w_{n_k} \in \mathcal{W}_{n_k}$ such that if
   (a) $a_k + b_k = q_{n_k}$
   (b) for each $J \in \mathcal{I}_{n_k}$

$$z^* \restriction [\min(J) - a_k, \max(J) - a_k) = w_{n_k} \restriction J$$

Then there is a measure ν on $\mathbb{K}$ such that $\phi^*(\nu)$ and μ are not mutually singular.

IDEA OF PROOF

3. $z \in X$ is (μ, Γ)-generic with (T, Γ)-name z^* and there are increasing positive sequences $\{n_k, a_k, b_k : k \in \mathbb{N}\}$ and words $w_{n_k} \in \mathcal{W}_{n_k}$ such that if

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Then there is a measure ν on $\mathbb{K}$ such that $\phi^*(\nu)$ and μ are not mutually singular.

Take a subsequence of the words w_{n_k} that are generic for μ. The fact that $\mathcal{I}_{n_k}$ is a test sequences says that the ν and μ measures of words agree up to a fixed proportion of both measures.

TWO JOBS

- Build the diffeomorphism T,
- Build the partition Γ.

BUILD THE DIFFEOMORPHISM

To my knowledge there is only one general method of constructing diffeomorphisms, the

Anosov-Katok method of approximation

Katok liked calling it the ABC method:

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Anosov-Katok method of approximation

Katok liked calling it the ABC method: Approximation by conjugacy.

- The circular operator captures the words generated by partitions using the (unskewed) ABC method.

GENERALITIES

- View the torus $\mathbb{T}^2$ as $[0,1) \times [0,1)$,

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$$\overline{\mathcal{R}}_\alpha(x,y) = (x+\alpha, y) \pmod 1$$

GENERALITIES

- View the torus $\mathbb{T}^2$ as $[0,1) \times [0,1)$,
- for an $\alpha \in [0,1)$ let $\overline{\mathcal{R}}_\alpha$ be the rotation

$$\overline{\mathcal{R}}_\alpha(x,y) = (x+\alpha, y) \quad (mod \quad 1)$$

- We will approximate the diffeomorphism T by periodic transformations of the form

$$T_n = H_n \overline{\mathcal{R}}_{\alpha_n} H_n^{-1}$$

where $H_n : [0,1) \times [0,1) \to [0,1) \times [0,1)$.

THE TRICK

- H_n is a composition of C^∞-diffeomorphisms h_n, $H_n = h_0 \circ h_1 \circ \ldots h_n$.

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so

- $T_{n+1} = h_0 \circ h_1 \circ \ldots h_{n+1} \circ \overline{\mathcal{R}}_{\alpha_{n+1}} \circ h_{n+1}^{-1} \circ \ldots h_1^{-1} \circ h_0^{-1}$

THE TRICK

- Make:

$$h_{n+1} \circ \overline{\mathcal{R}}_{\alpha_n} \circ h_{n+1}^{-1} = \overline{\mathcal{R}}_{\alpha_n}$$

- Then

$$T_n = H_n \circ h_{n+1} \circ \overline{\mathcal{R}}_{\alpha_n} \circ h_{n+1}^{-1} \circ H_n^{-1}.$$

- So ... choosing α_{n+1} sufficiently close to α_n makes H_{n+1} close to H_n.

TO DO

- choose the α_n's
- build the h_n's
- build a sequence of partitions Γ_n which converge universally to a partition Γ.

Anosov-Katok Numerology

Fix a *circular coefficient sequence* $\langle k_n, l_n : n \in \mathbb{N} \rangle$.

Let $p_0 = 0$ and $q_0 = 1$ and inductively set

$$q_{n+1} = k_n l_n {q_n}^2 \tag{3}$$

(thus $q_1 = k_0 l_0$) and take

$$p_{n+1} = p_n q_n k_n l_n + 1.$$

Then $(p_{n+1}, q_{n+1}) = 1$.

ANOSOV-KATOK NUMEROLOGY

Let

$$\alpha_n = \frac{p_n}{q_n}$$

Then

$$\alpha_{n+1} = \alpha_n + \frac{1}{k_n l_n q_n^2}$$

At stage n, let $j_i = p_n^{-1} i \pmod{q_n}$

HORIZONTAL PARTITIONS

- For $q \in \mathbb{N}$, let $\mathcal{I}_q$ be the partition of $[0,1)$ intervals of the form $[\frac{i}{q}, \frac{i+1}{q})$.

- The map $\overline{\mathcal{R}}_{\alpha_n}$ preserves the partition $[0,1) \times \mathcal{I}_q$.

How does $\overline{\mathcal{R}}_{\alpha_n}$ relate to $\overline{\mathcal{R}}_{\alpha_{n+1}}$?

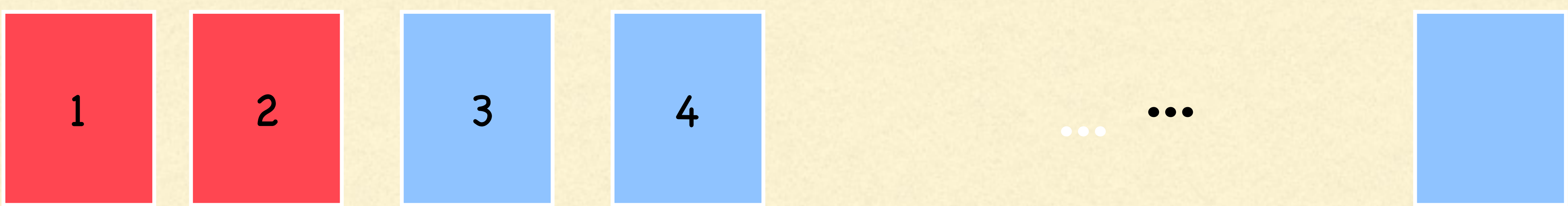
$$\alpha_n = \frac{p_n}{q_n}$$

Geometric ordering of 1/q_n intervals

width=1/q_n

translated here

Dynamical ordering of 1/q_n intervals



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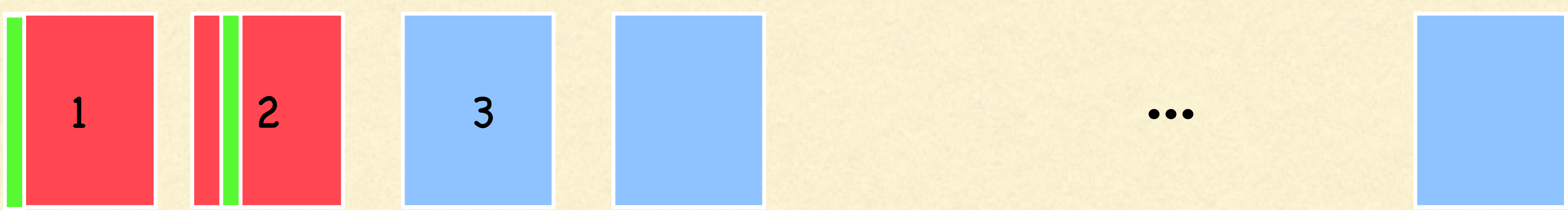

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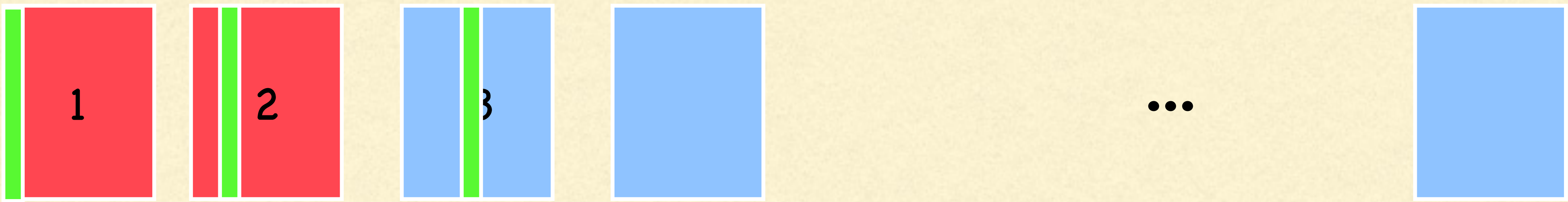

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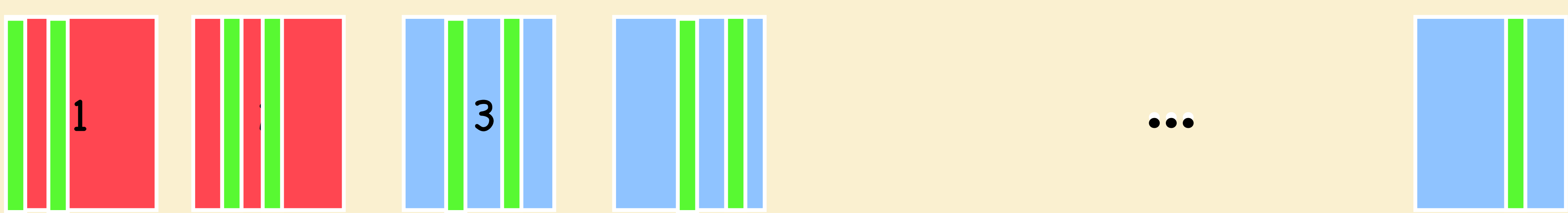

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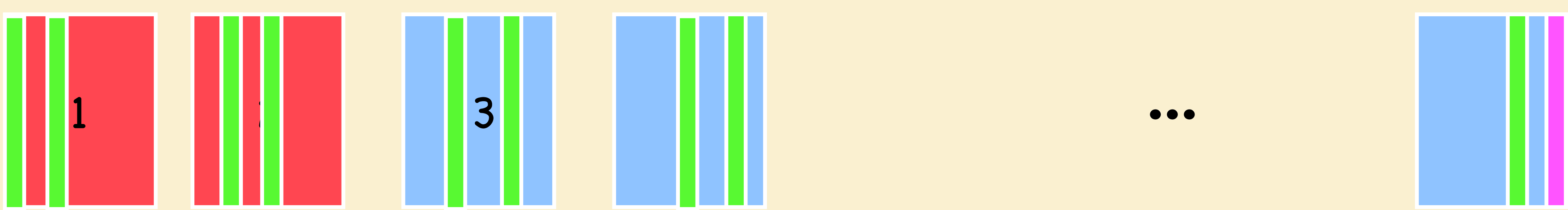

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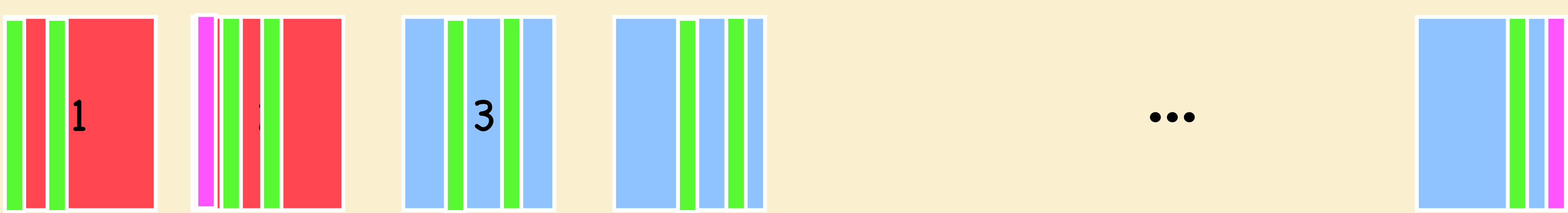

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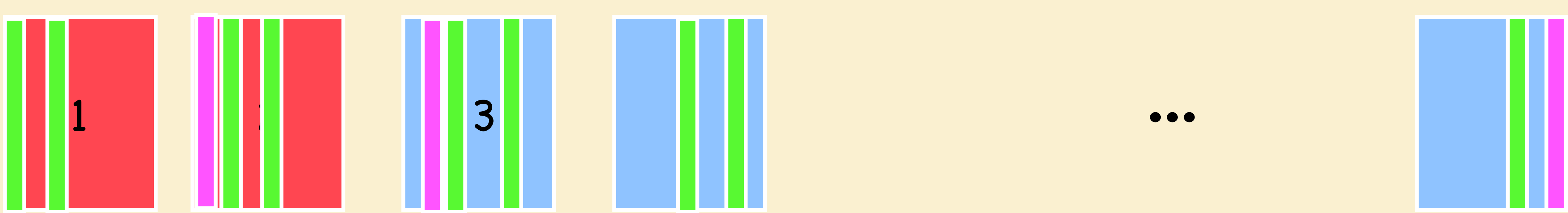

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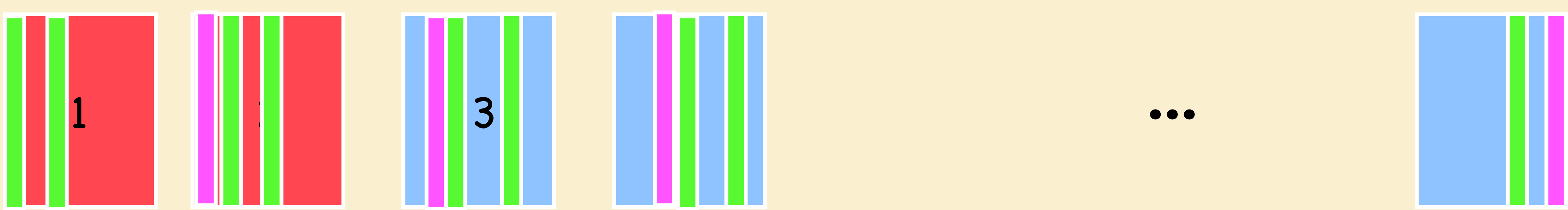

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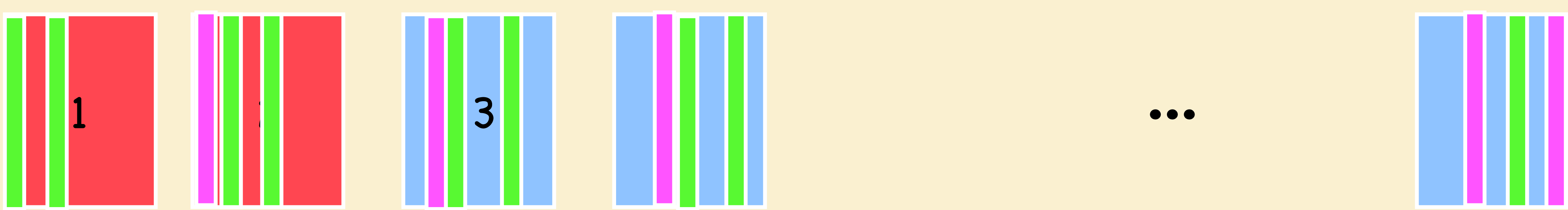

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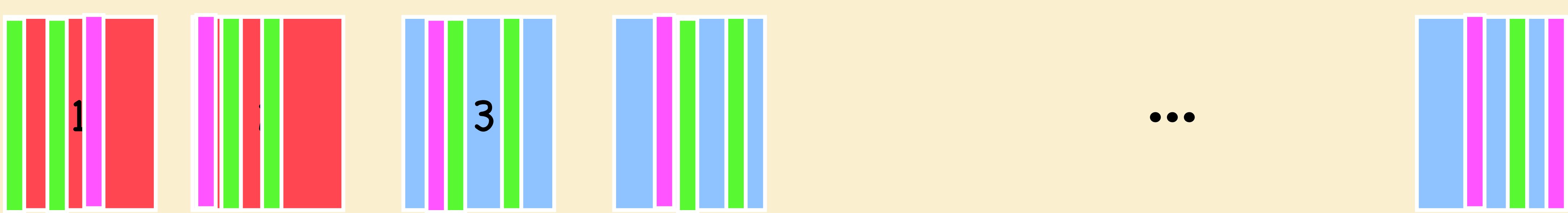


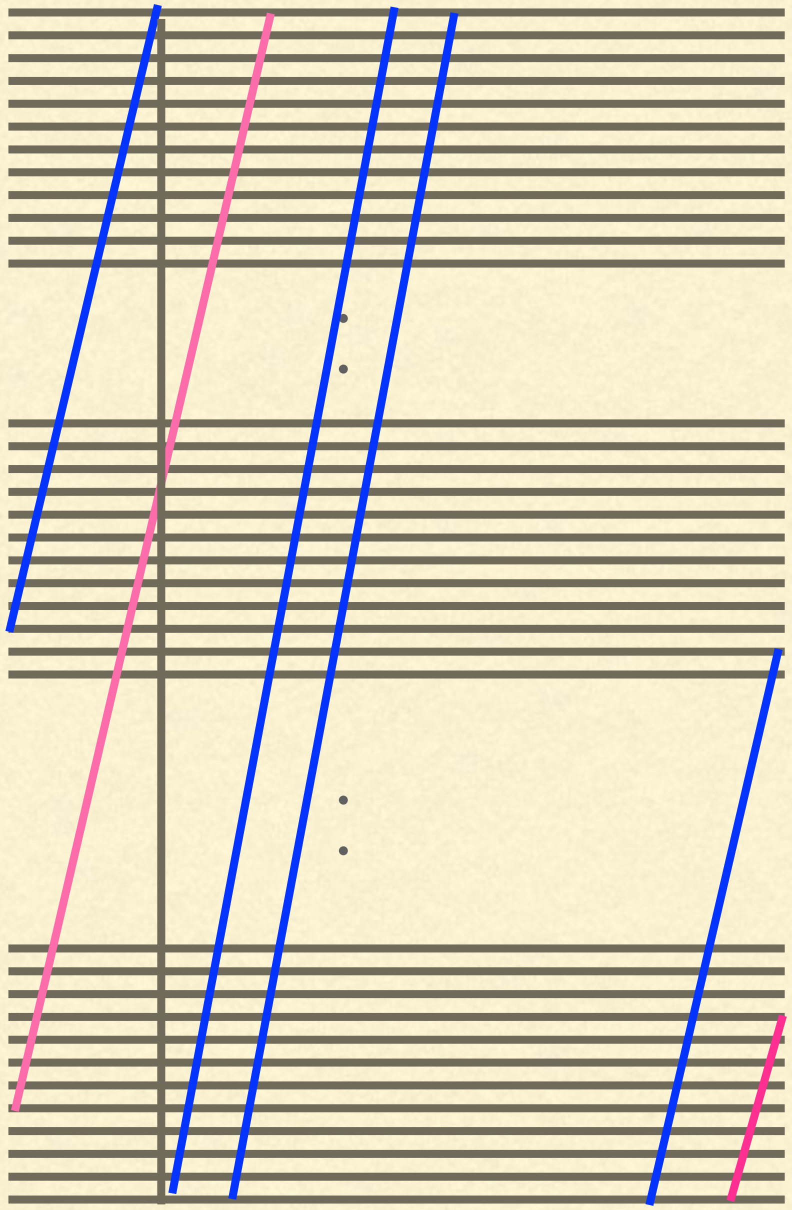
IN ENGLISH

- The $\overline{\mathcal{R}}_{\alpha_{n+1}}$ maps follow the $\overline{\mathcal{R}}_{\alpha_n}$ maps for stretches of length $k_n q_n l_n$, but then jump to the next geometric interval.
- If this is the i^{th} geometric interval it takes $q - j_i$ many steps to return to the interval $[0, 1/q_n)$.
- It crosses the intervals of the form $\frac{j}{k_n q_n}$ in q_n many steps before jumping to another interval.

THE PICTURE

The α_{n+1}-orbits go up diagonally through the dynamical ordering of the intervals of length $1/q_n$. They cross intervals of the form $j/q_n + i/k_n q_n$ at intervals j_i. In the formula for the circular words these correspond to the endings and the beginnings of the next word.

The lower part of the diagonal is a sequence of e's and the upper part is a sequence of b's.



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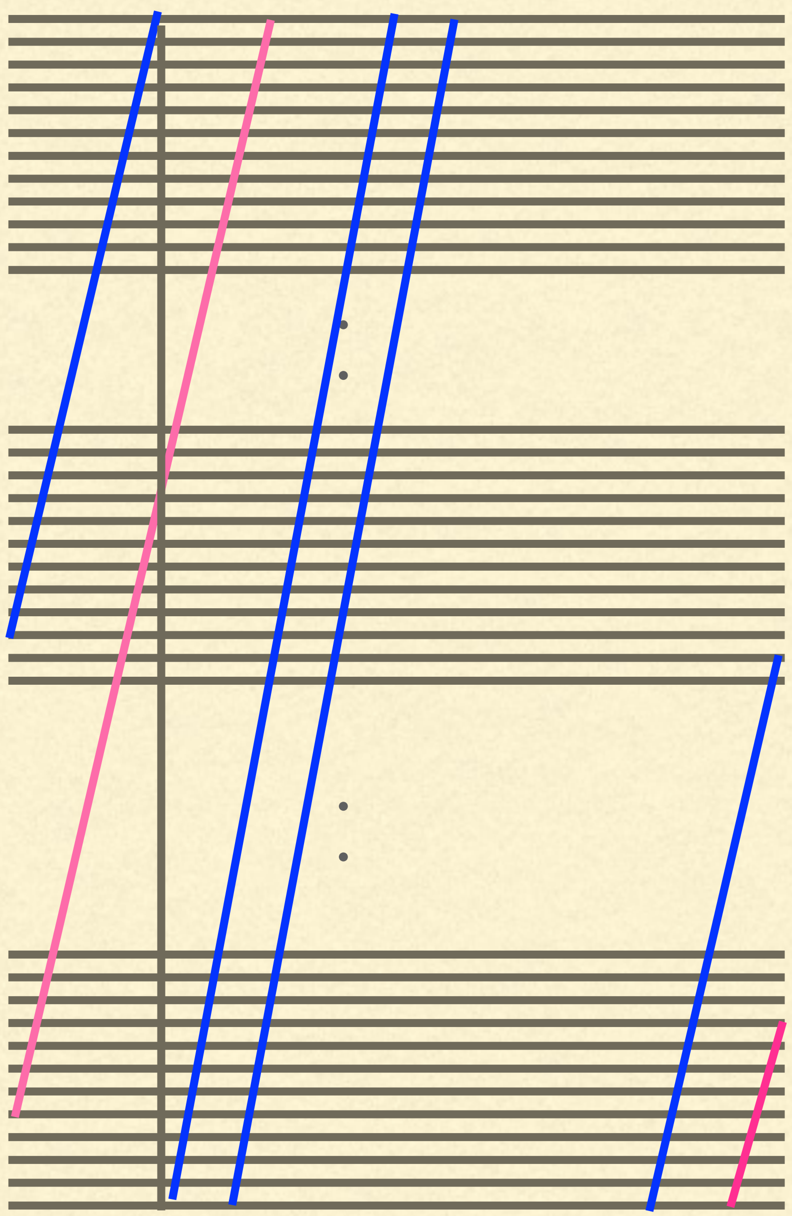
$$\mathcal{C}(w_0, w_1, w_2, \ldots w_{k-1}) = \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{q-j_i} w_j^{l-1} e^{j_i}).$$



THE PARTITIONS

There is an inductively defined sequence of partition $\langle \Gamma_n : n \in \mathbb{N} \rangle$ that converge to a partition Γ in a universally measurable sense.

At stage n:

1. There will be $\gamma_0 = 0 < \gamma_1 < \gamma_2 < \dots \gamma_{s_n} = 1$ such that points in the interval $[0, \frac{1}{q_n}) \times [\gamma_i, \gamma_{i+1})$ will have $(\overline{\mathcal{R}}_{\alpha_n}, \Gamma_n)$-name w_i for $w_i \in \mathcal{W}_n$.

2. h_{n+1} is defined on the region $[0, \frac{1}{q_n})$ and extended equivariantly with $\overline{\mathcal{R}}_{\alpha_n}$, so it commutes with $\overline{\mathcal{R}}_{\alpha_n}$.

3. h_{n+1} is defined so that the partition $\Gamma_{n+1} = h_{n+1}^{-1}(\Gamma_n)$ gives $\mathcal{W}_{n+1}$-names to a partition of $[0, 1)$.

How to build h_{n+1}

$$w = \prod_{i=0}^{q-1}\prod_{j=0}^{k-1}(b^{q-j_i} w_j^{l-1} e^{j_i}).$$

- We will build partitions Γ_n. Γ_{n+1} will be $h_{n+1}^{-1}\Gamma_n$.
- (Pre-skewing) Each $w \in \mathcal{W}_n$ will correspond to an interval of the form $[\gamma_i, \gamma_{i+1})$ and its orbit will be $\overline{\mathcal{R}}_{\alpha_n}([0, 1/q_n) \times [\gamma_i, \gamma_{i_1})$ which will have Γ_n name w.
- After skewing the orbit will be a sequence of adjacent parallelograms. The parallelograms are built so that generic points hit at least the parallelogram at least a fixed proportion of the time. (The "capturing measures" aspect.)

$$w = \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{q-j_i} w_j^{l-1} e^{j_i}).$$

We will write $\Gamma_n = \{P_i^n : i \in \Sigma\}$

- Since $\mathcal{W}_0 = \Sigma$, we assign the γ_i's so that if $\sigma_i \in \Sigma$ has measure δ then $\gamma_{i+1}^n - \gamma_i^n = \delta$.
- The partition Γ_0 puts the strip $[\gamma_i, \gamma_{i+1}) \times [0, 1)$ into P_i^0
- To pass to stage $n + 1$, if $[\gamma_i^{n+1}, \gamma_i^{n+1})$ is the interval assigned to w, then

  $$h_{n+1} : [j/k_n q_n, (j+1)/k_n q_n) \times [\gamma_i^{n+1}, \gamma_i^{n+1}) \to [0, 1/q) \times [\gamma_j^n, \gamma_{j+1}^n)$$

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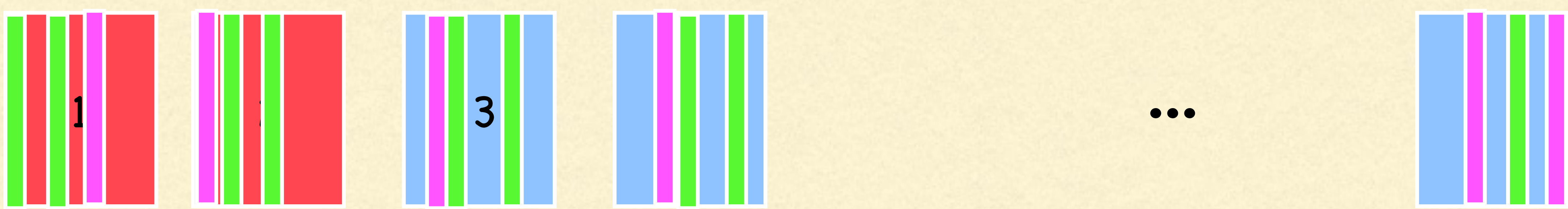
Dynamical ordering of 1/q_n intervals



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Dynamical ordering of 1/q_n intervals

Then tracking the names along an $\overline{\mathcal{R}}_{\alpha_{n+1}}$ trajectory of a point in $[\gamma_i^{n+1}, \gamma_{i+1}^{n+1} \times [0, 1/q_n)$ you get $\displaystyle\prod_{i=0}^{q-1}\prod_{j=0}^{k-1}(b^{q-j_i} w_j^{l-1} e^{j_i})$.

SUPPOSE WE HAVE THE RIGHT NAMES **IF** WE HAVE THESE MAPS

Remaining Problems

- Is there room inside $[0, 1/q) \times [\gamma_j^n, \gamma_{j+1}^n)$ to have the images of the map f all be disjoint?

  Solution: *Conservation of Mass Lemma*

- f can't possibly be smooth if it maps the exact intervals

  $$[j/k_n q_n, (j+1)/k_n q_n) \times [\gamma_i^{n+1}, \gamma_{i+1}^{n+1}$$

  into disjoint non-adjacent intervals.

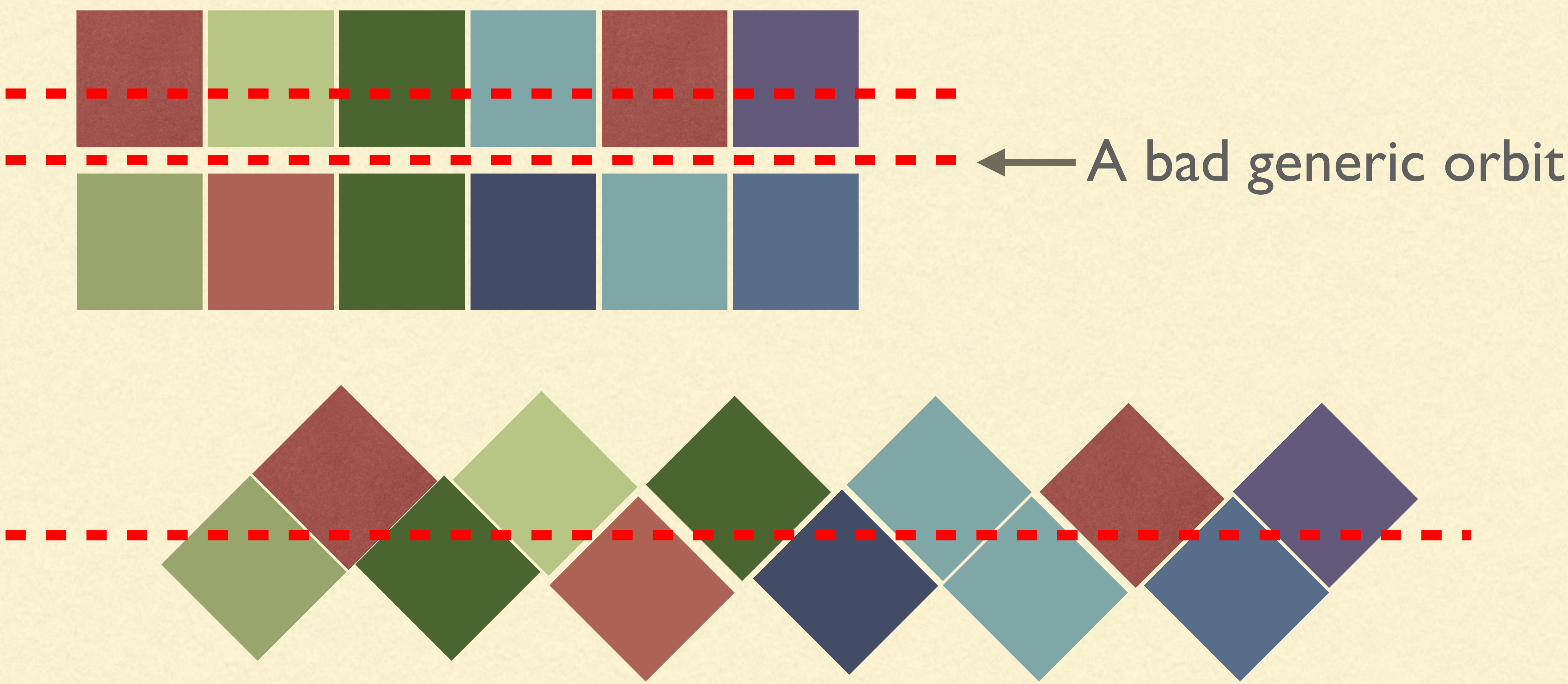
  Solution: Use permutation *Pasting Lemmas* to approximate f arbitrarily well with C^∞ maps.

- What happens along the seams of the rectangles

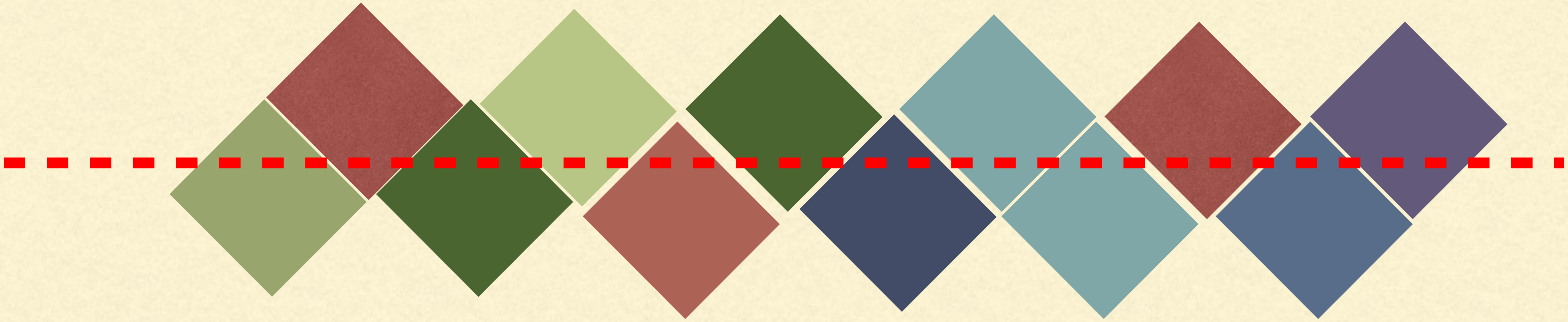
  $[j/k_n q_n, (j+1)/k_n q_n) \times [\gamma_i^{n+1}, \gamma_{i+1}^{n+1}, [j/k_n q_n, (j+1)/k_n q_n) \times [\gamma_{i+1}^{n+1}, \gamma_{i+2}^{n+1})$?

  There can be measures that concentrate on the errors between the C^∞ map and the f's.

TRICK WITH ROOTS IN FAYAD AND KATOK

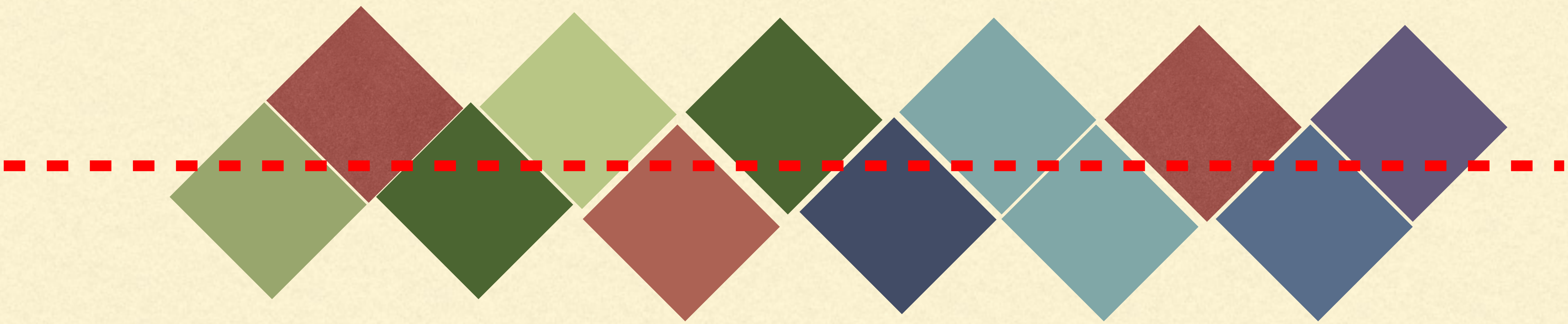


TRICK WITH ROOTS IN FAYAD AND KATOK



For every measure, a generic point traverses either the top word or the bottom word at least a fixed proportion of the time.

TRICK WITH ROOTS IN FAYAD AND KATOK



For every measure, a generic point traverses either the top word or the bottom word a fixed proportion of the time.

Hence there are test sequences for the words and one can apply the abstract discussion to capture all invariant measures.

THANK YOU!