Workshop

Structures Semester: Descriptive Set Theory & Dynamics

University of Warsaw
MIM
Faculty of Mathematics, Informatics and Mechanics

Banach Center
Institute of Mathematics PAS

Schedule

Monday	
9:30-10:30	**Matthew Foreman**, *Tutorial, part I*
10:30-11:00	coffee break
11:00-12:00	**Todor Tsankov**, *Tutorial, part I*
12:00-14:30	lunch break
14:30-15:30	**Anush Tserunyan**, *Tutorial, part I*
15:30-15:50	coffee break
15:50-16:20	**Konrad Wrobel**, *Measure equivalence and wreath products*
16:20-16:30	break
16:30-17:00	**Aranka Hrušková**, *Finite exchangeable graphs*
Tuesday	
9:30-10:30	**Todor Tsankov**, *Tutorial, part II*
10:30-11:00	coffee break
11:00-12:00	**Anush Tserunyan**, *Tutorial, part II*
12:00-14:30	lunch break
14:30-15:30	**Matthew Foreman**, *Tutorial, part II*
15:30-15:50	coffee break
15:50-16:20	**Shaun Allison**, *Polish automorphism groups involving S_∞*
16:20-16:30	break
16:30-17:00	**Colin Jahel**, *Invariant random structures and dynamics of...*
Wednesday	
9:30-10:30	**Anush Tserunyan**, *Tutorial, part III*
10:30-11:00	coffee break + poster session
11:00-12:00	**Yonatan Gutman**, *Tutorial, part I*

Thursday

9:30-10:30	**Todor Tsankov**, *Tutorial, part III*
10:30-11:00	coffee break
11:00-12:00	**Matthew Foreman**, *Tutorial, part III*
12:00-14:30	lunch break
14:30-15:30	**Yonatan Gutman**, *Tutorial, part II*
15:30-15:50	coffee break
15:50-16:20	**Forte Shinko**, *Hyperfiniteness of boundary actions on trees*
16:20-16:30	break
16:30-17:00	**Grigory Terlov**, *Nonamenable subforests of multi-ended...*

Friday

9:30-10:30	**Yonatan Gutman**, *Tutorial, part III*
10:30-10:50	coffeee break
10:50-11:20	**Felix Weilacher**, *Computable vs Descriptive Combinatorics of...*
11:20-11:30	break
11:30-12:00	**Riley Thornton**, *Limits and coloring for regular hypergraphs*

Abstracts

Tutorials

Matthew Foreman, *The simplex of measures invariant under diffeomorphisms*

The tutorial will concern two questions:

1. Which Choquet simplexes $\mathbb{K}$ can be realized as the simplex of invariant measures under C^{∞}-diffeomorphisms of compact manifolds?
2. Where does the equivalence relation of "conjugate by a measure preserving transformation" sit in the pre-partial ordering of Borel reducibility of equivalence relations?

The simplex of invariant measures In 1952, Oxtoby gave an example of a minimal flow with exactly two ergodic measures. In 1984, Williams extended this by showing that there are symbolic shifts with any finite number of ergodic measures.

Downarowicz, in his 1991 paper, showed that every Choquet simplex arises as the simplex of invariant measures for the class of minimal $0-1$ Toeplitz flows. This result is best-possible for symbolic systems, but leaves open the problem of invariant measures for diffeomorphisms.

In 2004, Fayad and Katok gave an example of a diffeomorphisms of the unit disk with exactly three ergodic invariant measures (hence the invariant measures are a simplex with three extreme points). Their method generalizes easily build diffeomorphisms whose simplex of invariant measures has exactly n extreme points for any finite n.

This tutorial focusses on a joint result with B. Weiss:

Theorem Let $\boldsymbol{K}$ be a Choquet simplex. Then there is a C^{∞}-diffeomorphism T of the 2-torus $\mathbb{T}^2$ such that $\boldsymbol{K}$ is the simplex of T-invariant measures. Moreover T can also be Lebesgue measure preserving.

This talk uses techniques in the proof of the joint results with B. Weiss that the measure-isomorphism relation for Lebesgue measure preserving diffeomorphisms of the torus is complete analytic. As part of that proof, there were two categories of symbolic shifts:

- Odometer based transformations
- Circular systems

These two categories are functorially isomorphism, by a functor that preserves the simplex of invariant measures. The circular systems can be realized as diffeomorphisms using the Ansov-Katok method of smooth realization. With B. Weiss, it is shown that the the realization can be done in a manner that preserves the simplex of invariant measures.

Every finite entropy measure preserving transformation with a non-trivial odometer factor can be realized as an odometer based system, in a manner that preserves the simplex of invariant measures. In particular each simplex $\boldsymbol{K}$ realized by a Toeplitz flow $\mathcal{T}$ constructed by Downarowicz can be realized as the simplex of invariant measures of an odometer based system $\mathcal{O}$. Using the functor, there is a circular system $\mathcal{C}$ having $\mathbb{K}$ as its simplex of invariant measures. Finally, $\mathcal{C}$ can be realized as a diffeomorphism of the torus having $\boldsymbol{K}$ as its simplex of invariant measures.

Is the isomorphism equivalence relation maximal? The second question addressed in the tutorial is where the equivalence relation "$\cong_{MPT}$" of conjugate-by-a-measure preserving transformation sits under the pre-partial ordering of Borel reducibility.

Since the group of measure preserving transformations is a Polish group, $\cong_{MPT}$ is induced by a Polish group action. The joint work with B. Weiss shows that $\cong_{MPT}$ is complete analytic. Later work of the lecturer shows that the equivalence relation of *Graph Isomorphism* is Borel reducible to $\cong_{MPT}$. This is the current state of knowledge.

A result of Sabok shows that relation of *affine homeomorphism* on the space of Choquet simplexes is maximal among equivalence relations induced by Polish group actions.

However there are conflicting conjectures as to where$\cong_{MPT}$ sits. Let $[T]$ be the full group of an ergodic measure preserving transformations and $\cong^{*}\subseteq [T]\times[T]$ be the relation of conjugacy by an element $\phi\in[T]$. It is known that $\cong^{*}$ is not maximal among Polish group actions.

Here are the two conflicting conjectures:

Conjecture 1 Sabok conjectures that the equivalence relation of *affine homeomorphism* of Choquet simplexes is Borel reducible to $\cong_{MPT}$, and hence that $\cong_{MPT}$ is maximal among Polish group actions.

Conjecture 2 Le Maître conjectures that $\cong_{MPT}$ is Borel reducible to $\cong^{*}$.

The tutorial will discuss what is known about these two conjectures.

Yonatan Gutman, *Nilspaces and their applications*

Consider the following celebrated results and developments of the last 20 years:

1. The Green-Tao theorem on arbitrarily long arithmetic progressions in the primes.
2. The Host-Kra nonconventional ergodic average L^2 convergence theorem.
3. The development of the so-called higher-order Fourier analysis by Gowers, Tao, Szegedy...

These and many other results in topological dynamics, ergodic theory, combinatorics, and number theory may be understood in the framework of *nilspaces* - certain compact spaces X equipped with closed collections of *cubes* $C^n(X) \subseteq X^{2^n}$, $n = 1, 2, \ldots$ satisfying three easy-to-state axioms.

The mini-course will be dedicated to the theory of nilspaces. More specifically I plan to discuss several concepts and proofs from the following articles:

Yonatan Gutman and Zhengxing Lian, Strictly ergodic distal models and a new approach to the Host-Kra factors. Journal of Functional Analysis 284 (2023), no. 4, 109779.

Yonatan Gutman, Freddie Manners and Péter Varjú, The structure theory of Nilspaces III: Inverse limit representations and topological dynamics. Advances in Mathematics 365 (2020), 107059.

Yonatan Gutman, Freddie Manners and Péter Varjú, The structure theory of Nilspaces I. Journal d'Analyse Mathématique 140 (2020), 299–369.

Yonatan Gutman, Freddie Manners and Péter Varjú, The structure theory of Nilspaces II: Representation as nilmanifolds. Transactions of the American Mathematical Society 371 (2019), 4951–4992.

Eli Glasner, Yonatan Gutman and XiangDong Ye, Higher order regionally proximal equivalence relations for general minimal group actions. Advances in Mathematics 333 (2018), 1004–1041.

Todor Tsankov, *Invariant and quasi-invariant measures for actions of oligomorphic groups*

Measure-preserving actions of infinite permutation groups arise naturally when one studies random processes with certain symmetry conditions. If the group is large (for example, oligomorphic), one can often obtain strong structure results for the corresponding processes. In this minicourse, I will explain some of these results and the main tools (coming from representation theory) used to obtain them. It turns out that they also help to describe quasi-measure-preserving actions of oligomorphic groups. Parts of the tutorial are based on joint work with Colin Jahel.

Anush Tserunyan, *The geography of amenable subrelations of acyclic graphs and beyond*

The Adams dichotomy for pmp equivalence relations characterizes the amenable ones among treeable relations. In a joint work with Robin Tucker-Drob, we generalize this to the measure class preserving setting, where no theory of cost is at hand. Instead we exploit end selection, revealing the intrinsic reason why the particular ends are selected. I will explain this result and its consequences, e.g., uniqueness of maximal amenable subrelation, and give samples of techniques involved, including mass transport, tree geometry (minimal convex sections, Helly's theorem), and tree dynamics (paddle-ball lemma, Carrier–Ghys theorems). If time permits, I will also discuss a sequel to this work, joint with Ruiyuan Chen and Grigory Terlov, concerning multi-ended graphs.

Talks

Shaun Allison, *Polish automorphism groups involving S_∞*

A Polish automorphism group, also known as a non-Archimedean Polish group, is the automorphism group of a countable discrete structure in a countable language. The Polish group S_∞ is the "strongest" such group, being the automorphism group of a countably-infinite set with empty language. A Polish group G involves a Polish group H iff there is a continuous surjective homomorphism from a closed subgroup of G onto H. By Mackey-Hjorth, any Borel action of S_∞ can be "emulated" by a Borel action of any Polish group involving S_∞. We give several equivalent conditions for a Polish automorphism group to involve S_∞ that are all of a significantly different nature, including a condition involving a rank function, a weakening of disjoint amalgamation, and classification of $=^+$. This gives greater clarity into the space of Polish groups that are too strong to have a compatible complete left-invariant metric, but not strong enough to involve S_∞. Time permitting, we discuss Knight's model, which is essentially the simplest such group.

Aranka Hrušková, *Finite exchangeable graphs*

De Finetti's theorem says that the extreme points of the set of exchangeable probability distributions on binary sequences are the random sequences in which the individual entries are iids. Every exchangeable probability distribution can then be obtained by integrating over these product measures with respect to a probability measure. It is known that while the product measures are not the extreme points of the set of exchangeable probability distributions on *finite* binary sequences, every exchangeable measure can still be obtained as an integral over these, but with respect to a *signed* measure. We prove the higher dimensional analogue of this statement, in which binary sequences become graphs and the role of the product measures is played by graphons. This is work in progress, joint with Peter Orbanz.

Colin Jahel, *Invariant random structures and dynamics of non-archimedean Polish groups*

Joint with Matthieu Jospeh. In their paper of 2012, Ackerman Freer and Patel greatly expanded the class of structures that can reasonably be called "random" to include all homogeneous structures with no algebraicity. Their results states that for a structure M, there exists a S_∞ invariant measure on the space of structures, a.s. isomorphic to M. We study actions of closed subgroups of S_∞ on the space of structures, which are related to their work but yields some new suprising and interesting results. Our goal is not to get existence of such measures but to study how those measures behave dynamically. We get in particular a full description of all ergodic action of a great number of subgroups of S_∞. As a corollary to this, we also get the topological simplicity of so-called de Finetti groups.

Forte Shinko, *Hyperfiniteness of boundary actions on trees*

Let G be a countable group of automorphisms of a countable tree T. We show that assuming a mild form of acylindricity, the induced action of G on the boundary of T is hyperfinite, in both the Borel and measurable contexts. This is joint with Kunnawalkam Elayavalli, Oyakawa and Spaas.

Grigory Terlov, *Nonamenable subforests of multi-ended quasi-pmp graphs*

In the last thirty years, amenability of probability measure preserving (pmp) Borel actions of countable groups has been well understood, largely due to the theory of cost available for pmp countable Borel equivalence relations. On the other hand, very little is known in the quasi-pmp (measure-class preserving) setting, where cost does not yield desirable results. Moreover, since nonamenable groups, such as F_2, can have free amenable quasi-pmp actions, the behavior in this setting has been regarded as particularly mysterious. In this talk, I will present a construction of a nonamenable subforest of multi-ended quasi-pmp Borel graphs. This, together with a result of Tserunyan and Tucker-Drob, witnesses nonamenablility of quasi-pmp actions, whose orbit equivalence relations admit such graphings. The main technique is a weighted cycle-cutting algorithm, which yields a weight-maximal spanning forest. We also introduce a random version of this forest, which generalizes the Free Minimal Spanning Forest, to capture nonunimodularity in the context of percolation theory. This is joint work with Ruiyuan Chen and Anush Tserunyan.

Riley Thornton, *Limits and coloring for regular hypergraphs*

The theory of weak containment (or equivalently local-global convergence) gives a nice pictures of sequences of bounded degree graphs with connections to ergodic theory and probability. I will report on progress generalizing this picture to hypergraphs and give an application to CSPs.

Felix Weilacher, Computable vs Descriptive Combinatorics of Local Problems on Trees

We study the position of the computable setting in the "common theory of locality" developed by Brandt et. al. for local problems on Δ-regular trees $\Delta \in \omega$. We show that such a problem admits a computable solution on every highly computable regular forest if and only if it admits a Baire measurable solution on every Borel Δ-regular forest. Time permitting, we also discuss broader parallels between graph combinatorics in the Computable and Baire measurable settings.

Konrad Wrobel, *Measure equivalence and wreath products*

Let F be a nonabelian free group. We show that if G and H are nontrivial measure equivalent groups, then the wreath products $G \wr F$ and $H \wr F$ are orbit equivalent. This is joint work with Robin Tucker-Drob.